1. $y' - 3y = e^{2t}\ ;\ y(0) = 1$

$$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$$

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s-2} \Rightarrow (s-3)Y(s) = \frac{1}{s-2} + 1 \Rightarrow Y(s) = \frac{s-1}{(s-2)(s-3)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s-3)}\right\} \Rightarrow \boxed{y(t) = -e^{2t} + 2e^{3t}}$$

$$\frac{s-1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

$$\left.\frac{s-1}{s-3}\right|_{s=2} = A \Rightarrow A = -1 \ ; \quad \left.\frac{s-1}{s-2}\right|_{s=3} = B \Rightarrow B = 2$$

2. $y'' - 6y' + 9y = t^2 e^{3t}\ ;\ y(0) = 2,\ y'(0) = 6$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9Y(s) = \mathcal{L}\{t^2 e^{3t}\}$$

$$\mathcal{L}\{t^2\}\big|_{s \to s-3} = \frac{2}{(s-3)^3}$$

$$s^2 Y(s) - s\,y(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s)$$

$$(s^2 - 6s + 9)Y(s) - 2s - 6 + 12$$

$$(s-3)^2 Y(s) - 2s + 6$$

$$(s-3)^2 Y(s) = \frac{2}{(s-3)^3} + 2s - 6 \Rightarrow Y(s) = \frac{2}{(s-3)^5} + \frac{2}{(s-3)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s^5}\right|_{s \to s-3}\right\} + 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s}\right|_{s \to s-3}\right\}$$

$$= 2e^{3t}\frac{1}{4!}t^4 + 2e^{3t}$$

$$\boxed{y(t) = \frac{1}{12}t^4 e^{3t} + 2e^{3t}}$$

③ $y''+4y'+6y=1+e^{-t}$; $y(0)=0$; $y'(0)=0$.

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y'\}+6Y(s)=\mathcal{L}\{1\}+\mathcal{L}\{e^{-t}\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{0}-\underbrace{y'(0)}_{0}+4sY(s)-4\underbrace{y(0)}_{0}+6Y(s)=\frac{1}{s}+\frac{1}{s+1}$$

$$(s^2+4s+6)Y(s)=\frac{1}{s}+\frac{1}{s+1} \Rightarrow \boxed{Y(s)=\frac{2s+1}{s(s+1)(s^2+4s+6)}}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{6}+\frac{1}{3}e^{-t}\cdot t-\frac{1}{2}e^{-2t}\cos(\sqrt{2}t)-\frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t).$$

$$\frac{2s+1}{s(s+1)(s^2+4s+6)}=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+4s+6}$$

$$\left.\frac{2s+1}{(s+1)(s^2+4s+6)}\right|_{s=0}=A \Rightarrow A=1/6 \ ; \quad \left.\frac{2s+1}{s(s^2+4s+6)}\right|_{s=-1}=B \Rightarrow B=1/3$$

$$\Rightarrow \frac{Cs+D}{s^2+4s+6}=\frac{2s+1}{s(s+1)(s^2+4s+6)}-\frac{1}{6s}-\frac{1}{3(s+1)}$$

$$=\frac{12s+6-(s+1)(s^2+4s+6)-2s(s^2+4s+6)}{6s(s+1)(s^2+4s+6)}$$

$$=\frac{-3s-10}{6(s^2+4s+6)}$$

$$-12s-6+(3s+1)(s^2+4s+6)$$

$$\cancel{-12s}\cancel{-6}+3s^3+12s^2+18s+s^2+4s+\cancel{6}$$

$$3s^3+13s^2+10s$$

$$s(3s^2+13s+10)$$

$$s(s+1)(3s+10)$$

$$\Rightarrow Cs+D=\frac{1}{2}s+\frac{5}{3} \Rightarrow C=-1/2 \quad D=-5/3$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+6}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+2}\right\}=e^{-2t}\cos(\sqrt{2}t)-\sqrt{2}e^{-2t}\sin(\sqrt{2}t)$$

$$\mathcal{L}^{-1}\left\{\frac{-1/2s-5/3}{s^2+4s+6}\right\}=-\frac{1}{2}e^{-2t}\cos(\sqrt{2}t)+\underbrace{\frac{1}{\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)-\frac{5}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)}_{-\frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t)}$$

④ $y'' - y' = e^t \cos t \; ; \; y(0) = y'(0) = 0$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} - sY(s) + \underbrace{y(0)}_{0} = \mathcal{L}\{e^t \cos t\} = \mathcal{L}\{\cos t\}\Big|_{s \to s-1} = \frac{s-1}{(s-1)^2+1}$$

$$\Rightarrow (s^2 - s)\,Y(s) = \frac{s-1}{(s-1)^2+1} \Rightarrow Y(s) = \frac{1}{s\left((s-1)^2+1\right)} \Rightarrow \boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-2s+2)}\right\}}$$

$$\frac{1}{s(s^2-2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2-2s+2} \Rightarrow \frac{1}{s^2-2s+2}\Big|_{s=0} = A \Rightarrow A = 1/2$$

$$\Rightarrow \frac{Bs+C}{s^2-2s+2} = \frac{1}{s(s^2-2s+2)} - \frac{1}{2s} = \frac{\cancel{2} - s^2 + 2s - \cancel{2}}{2s(s^2-2s+2)} = \frac{-s+2}{2(s^2-2s+2)}$$

$$\Rightarrow B = -1/2 \; ; \; C = 1$$

$$\Rightarrow y(t) = \frac{1}{2} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{2} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s-1+1}{(s-1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{2} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s-1}\right\} = \boxed{\frac{1}{2} - \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t}$$

⑤ $y^{(4)} - y = 0; \quad y(0)=0, \; y'(0)=1, \; y''(0)=y'''(0)=0$

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0$$

$$\left[s^4 Y(s) - s^3 \underbrace{y(0)}_{0} - s^2 \underbrace{y'(0)}_{1} - s\underbrace{y''(0)}_{0} - \underbrace{y'''(0)}_{0}\right] - Y(s) = 0$$

$$(s^4-1)Y(s) = s^2 \Rightarrow \boxed{Y(s) = \frac{s^2}{s^4-1}} \Rightarrow \boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{s^2}{s^4-1}\right\}}$$

$$\frac{s^2}{\underbrace{s^4-1}_{(s^2-1)(s^2+1)}} = \frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$s^2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1)$$

$s=-1$: $\quad 1 = B(-2)(2) = -4B \Rightarrow \boxed{B = -1/4}$

$s=1$: $\quad 1 = 4A \Rightarrow \boxed{A = 1/4}$

$s=0$: $\quad 0 = A - B - D = \frac{1}{2} - D \Rightarrow \boxed{D = 1/2}$

$s=2$: $\quad 4 = 15A + 5B + 6C + 3D$

$$= \frac{15-5}{4} + \frac{3}{2} + 6C$$

$$= \underbrace{\frac{5}{2} + \frac{3}{2}}_{4} + 6C \Rightarrow \boxed{C=0}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} + \frac{1/2}{s^2+1}\right\}$$

$$\boxed{= \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{1}{2}\sin(t)}$$

$$= \frac{1}{2}\sinh(t) + \frac{1}{2}\sin(t).$$