Math 308 - Differential Equations **Homework 10**
Section 501
Texas A&M, Spring 2022

The Inverse Laplace Transform

Part I:

Find the inverse Laplace transforms below. (See next pages for useful formulas.)

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\}$

4. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

5. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\}$

6. $\mathscr{L}^{-1}\left\{\frac{1}{s^2-16}\right\}$

7. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\}$

8. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$

9. $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$

11. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\}$

12. $\mathscr{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$

Part II:

Solve the IVPs below *using the Laplace transform*:

1. $y'-3y=e^{2t}$; $y(0)=1$.

2. $y''-6y'+9y=t^2e^{3t}$; $y(0)=2, y'(0)=6$.

3. $y''+4y'+6y=1+e^{-t}$; $y(0)=y'(0)=0$.

4. $y''-y'=e^t\cos t$; $y(0)=y'(0)=0$.

5. $y^{(4)}-y=0$; $y(0)=0$; $y'(0)=1$; $y''(0)=y'''(0)=0$.

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ \ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ \ s > 0$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a};\ \ s > a$

$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2};\ \ s > 0$

$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2};\ \ s > 0$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ \ s > 0,\ a > 0.$

$\mathscr{L}\{\sinh(at)\} = \frac{k}{s^2-a^2};\ \ s > |a|$

$\mathscr{L}\{\cosh(at)\} = \frac{s}{s^2-a^2};\ \ s > |a|$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) = \mathscr{L}\{f(t)\}|_{s\to s-a}$$

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-a}\right\} = e^{at}f(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0) \\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\ \vdots \\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$