

## Unit Step Functions & Periodic Functions

$$\textcircled{1} \mathcal{L}\{(t-1)u_1(t)\} = e^{-s} \mathcal{L}\{t\} = \frac{1}{s^2} e^{-s}$$

$$\textcircled{2} \mathcal{L}\{t u_2(t)\} = \mathcal{L}\{(t-2+2)u_2(t)\} = \mathcal{L}\{(t-2)u_2(t)\} + 2\mathcal{L}\{u_2(t)\} \\ = e^{-2s} \mathcal{L}\{t\} + 2 \frac{e^{-2s}}{s} = e^{-2s} \left( \frac{1}{s^2} + \frac{2}{s} \right)$$

$$\textcircled{3} \mathcal{L}\{\cos(2t)u_{\pi}(t)\} = \mathcal{L}\{\cos(2t-2\pi)u_{\pi}(t)\} = \mathcal{L}\{\cos(2(t-\pi))u_{\pi}(t)\} \\ = e^{-\pi s} \mathcal{L}\{\cos(2t)\} = e^{-\pi s} \frac{s}{s^2+4}$$

$$\textcircled{4} \mathcal{L}\{(t-1)^3 e^{t-1} u_1(t)\} = e^{-s} \mathcal{L}\{t^3 e^t\} = e^{-s} \left( \mathcal{L}\{t^3\} \Big|_{s \rightarrow s-1} \right) = e^{-s} \left( \frac{3!}{s^4} \Big|_{s \rightarrow s-1} \right) \\ = \frac{6e^{-s}}{(s-1)^4}$$

$$\textcircled{5} \mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^3} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^3} \right\} \Big|_{t \rightarrow t-2} u_2(t) = \left( \frac{1}{2!} t^2 \Big|_{t \rightarrow t-2} \right) u_2(t) = \frac{(t-2)^2}{2} u_2(t)$$

$$\textcircled{6} \mathcal{L}^{-1}\left\{ \frac{e^{-\pi s}}{s^2+1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} \Big|_{t \rightarrow t-\pi} u_{\pi}(t) = \sin(t-\pi) u_{\pi}(t) = -\sin t \cdot u_{\pi}(t)$$

$$\textcircled{7} \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s(s+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\} \Big|_{t \rightarrow t-1} u_1(t) = (1 - e^{-t+1}) u_1(t)$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1} \right\} = 1 - e^{-t}$$

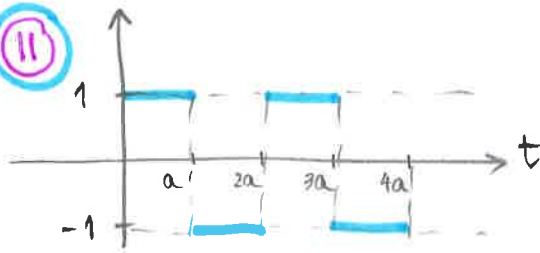
$$\textcircled{8} \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} e^{-3s} \right\} = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \right\} \Big|_{t \rightarrow t-3} u_3(t) = \cos(2t-6) u_3(t)$$

$$\textcircled{9} f(t) = \begin{cases} 2, & 0 \leq t < 3 \\ -2, & t \geq 3 \end{cases} \quad f(t) = 2(1 - u_3(t)) - 2u_3(t) = 2 - 4u_3(t) \\ \mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{4}{s} e^{-3s}$$

$$\textcircled{10} f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t^2, & t \geq 1 \end{cases} \quad f(t) = t^2 u_1(t)$$

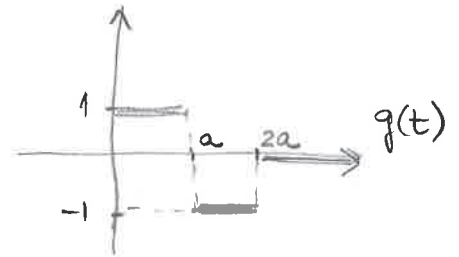
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u_1(t)\} = \mathcal{L}\{[(t-1)+1]^2 u_1(t)\} = \mathcal{L}\{[(t-1)^2 + 2(t-1) + 1] u_1(t)\} \\ = e^{-s} \mathcal{L}\{t^2\} + 2e^{-s} \mathcal{L}\{t\} + \mathcal{L}\{u_1(t)\} = e^{-s} \left( \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)$$

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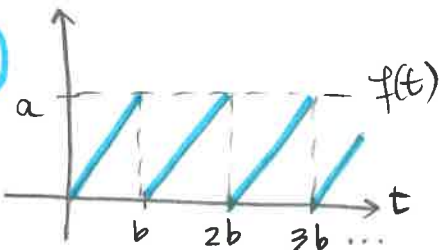
Period  $T=2a$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}} \underbrace{\int_0^{2a} e^{-st} f(t) dt}_{\mathcal{L}\{g(t)\}} \\ &= \frac{1}{1-e^{-2as}} \left( \frac{1-2e^{-as}+e^{-2as}}{s} \right) \\ &= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})} \end{aligned}$$



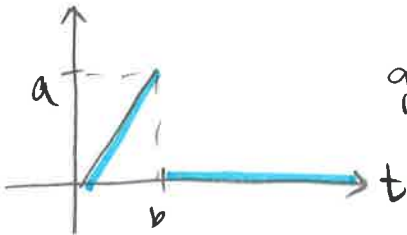
$$\begin{aligned} g(t) &= (1-u_a(t)) - 1(u_a(t) - u_{2a}(t)) \\ &= 1 - 2u_a(t) + u_{2a}(t) \\ \mathcal{L}\{g(t)\} &= \frac{1}{s} - 2\frac{e^{-as}}{s} + \frac{e^{-2as}}{s} \end{aligned}$$

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Period  $T=b$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-bs}} \underbrace{\int_0^b e^{-st} f(t) dt}_{\mathcal{L}\{g(t)\}}$$

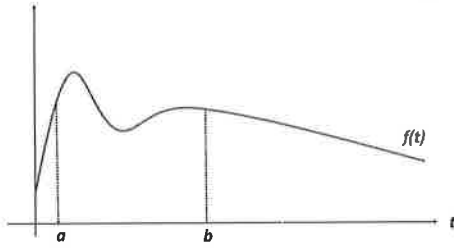


$$g(t) = \frac{a}{b} t (1-u_b(t))$$

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \frac{a}{b} \mathcal{L}\{t - t u_b(t)\} \\ &= \frac{a}{b} \left( \frac{1}{s^2} - \mathcal{L}\{(t-b)u_b(t)\} - b \mathcal{L}\{u_b(t)\} \right) \\ &= \frac{a}{b} \left( \frac{1}{s^2} - e^{-bs} \frac{1}{s^2} - \frac{b e^{-bs}}{s} \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-bs}} \frac{a}{b} e^{-bs} \left( e^{bs} \frac{1}{s^2} - \frac{1}{s^2} - \frac{b}{s} \right) \\ &= \frac{1}{e^{bs}-1} \frac{a}{b} \frac{e^{bs}-1-bs}{s^2} = \boxed{\frac{a(e^{bs}-1-bs)}{bs^2(e^{bs}-1)}} \end{aligned}$$

13. Consider the function  $f(t)$ , for  $t \in [0, \infty)$ , graphed below, and points  $a, b \in [0, \infty)$ .



Match each of the following graphs (obtained by various translations and "turning off" of the graph of  $f$ ) with the expressions below:

①.  $f(t)(1 - u_b(t))$

②.  $f(t)(u_a(t) - u_b(t))$

③.  $f(t)(1 - u_a(t))$

④.  $f(t)u_a(t)$

⑤.  $f(t-a)(u_a(t) - u_b(t))$

⑥.  $f(t-b)u_b(t)$

