Math 308 - Differential Equations **Homework 11**

Section 501

Texas A&M University, Spring 2022.

Laplace Transform:
Unit Step Functions and Periodic Functions

Compute the following:

1. $\mathscr{L}\{(t-1)u_1(t)\}$

2. $\mathscr{L}\{tu_2(t)\}$

3. $\mathscr{L}\{\cos(2t)u_\pi(t)\}$

4. $\mathscr{L}\{(t-1)^3e^{t-1}u_1(t)\}$

5. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$

6. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

7. $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$

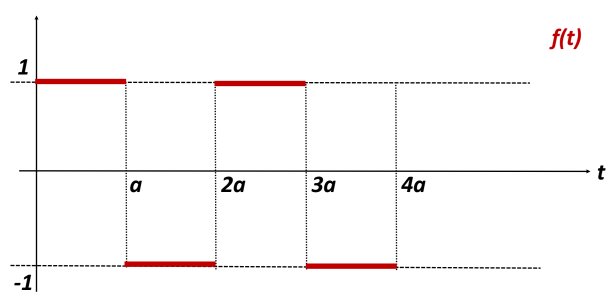
8. $\mathscr{L}^{-1}\left\{\dfrac{e^{-3s}s}{s^2+4}\right\}$

Express the following functions in terms of unit step functions and compute their Laplace transforms:

9. $f(t)=\begin{cases} 2 & ,0\le t<3 \\ -2 & ,t\ge 3. \end{cases}$

10. $f(t)=\begin{cases} 0 & ,0\le t<1 \\ t^2 & ,t\ge 1. \end{cases}$

11. The following graph describes the periodic function $f(t)$. Find $\mathscr{L}\{f(t)\}$.



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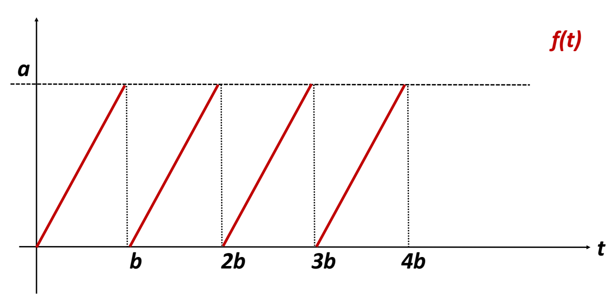
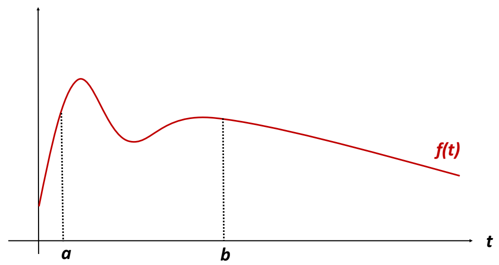
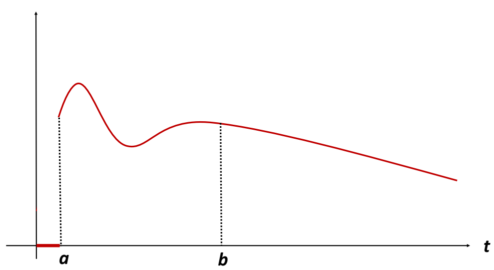


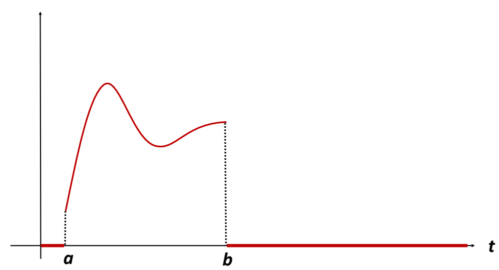
13. Consider the function $f(t)$, for $t \in [0, \infty)$, graphed below, and points $a, b \in [0, \infty)$.

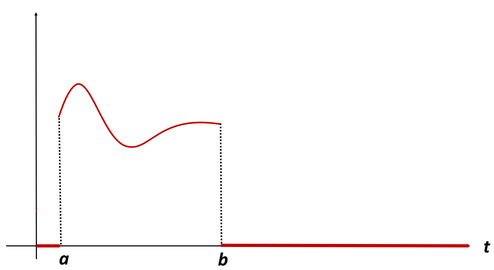


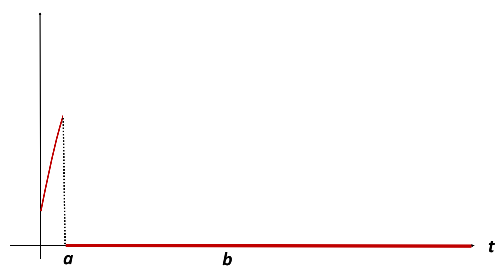
Match each of the following graphs (obtained by various translations and "turning off" of the graph of f) with the expressions below:

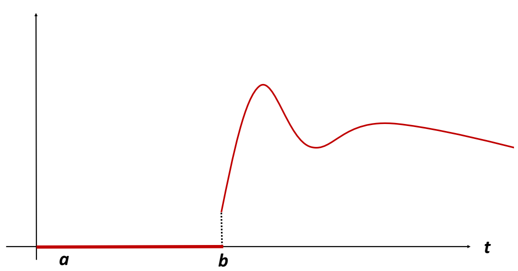
1. $f(t)(1 - u_b(t))$
2. $f(t)(u_a(t) - u_b(t))$
3. $f(t)(1 - u_a(t))$
4. $f(t)u_a(t)$
5. $f(t - a)(u_a(t) - u_b(t))$
6. $f(t - b)u_b(t)$

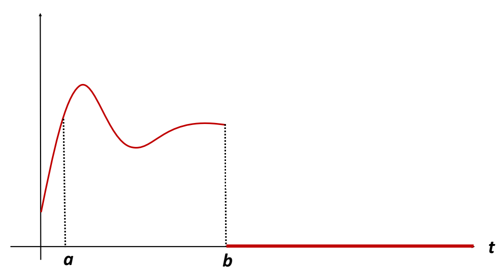












Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a};\ s > a$

$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2};\ s > 0$

$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2};\ s > 0$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0;\ a > 0.$

$\mathscr{L}\{\sinh(at)\} = \frac{a}{s^2-a^2};\ s > |a|$

$\mathscr{L}\{\cosh(at)\} = \frac{s}{s^2-a^2};\ s > |a|$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) = \mathscr{L}\{f(t)\}|_{s\to s-a}$$

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-a}\right\} = e^{at}\mathscr{L}^{-1}\{F(s)\} = e^{at}f(t)$$

Translation Theorem II:

$$\mathscr{L}\{f(t-a)u_a(t)\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}\,u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\ \ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$