Homework 12: Solutions

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$; $k \neq 0$

$$= \mathcal{L}^{-1}\{F(s)\cdot G(s)\} \quad \text{where} \quad F(s) = G(s) = \frac{1}{s^2+k^2}$$

$$= (f*g)(t) \quad \text{where} \quad f(t) = g(t) = \frac{1}{k}\sin(kt)$$

$$(f*g)(t) = \int_0^t \frac{1}{k}\sin(k(t-y))\cdot\frac{1}{k}\sin(ky)\,dy$$

$$= \frac{1}{k^2}\int_0^t \sin(kt-ky)\cdot\sin(ky)\,dy$$

$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

$$= \frac{1}{k^2}\int_0^t \frac{1}{2}\left(\cos(kt-2ky)-\cos(kt)\right)dy$$

$$= \frac{1}{2k^2}\left(\int_0^t \cos(kt-2ky)\,dy - \int_0^t \cos(kt)\,dy\right)$$

$$= \frac{1}{2k^2}\left(\frac{-1}{2k}\sin(kt-2ky)\Big|_{y=0}^{t} - \cos(kt)\cdot y\Big|_{y=0}^{t}\right)$$

$$= \frac{1}{2k^2}\left(\frac{-1}{2k}\underbrace{\sin(-kt)}_{=-\sin(kt)} + \frac{1}{2k}\sin(kt) - \cos(kt)\cdot t\right)$$

$$= \frac{1}{2k^2}\left(\frac{1}{k}\sin(kt) - \cos(kt)\cdot t\right) = \boxed{\frac{1}{2k^3}\sin(kt) - \frac{t}{2k^2}\cos(kt)}$$

2. $\mathcal{L}^{-1}\left\{\dfrac{G(s)}{(s-1)^2+1}\right\} = \mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$

where $F(s) = \dfrac{1}{(s-1)^2+1}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s-1}\right\}$$

$$= e^t\sin(t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{G(s)}{(s-1)^2+1}\right\} = (f*g)(t)$$

where $f(t) = e^t\sin(t)$

③ $y'' + K^2 y = g(t)$; $y(0) = \alpha$, $y'(0) = \beta$

$K \neq 0$, α, β are constants

$g(t)$ = piecewise continuous, of exponential order

$$s^2 Y(s) - s\,y(0) - y'(0) + K^2 Y(s) = G(s)$$

$$(s^2 + K^2) Y(s) = G(s) + \alpha s + \beta$$

$$Y(s) = \frac{G(s)}{s^2 + K^2} + \frac{\alpha s + \beta}{s^2 + K^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{G(s) \cdot \frac{1}{s^2 + K^2}\right\} + \alpha\,\mathcal{L}^{-1}\left\{\frac{s}{s^2 + K^2}\right\} + \beta\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + K^2}\right\}$$

$(f * g)(t)$, where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + K^2}\right\} = \frac{1}{K}\sin(Kt)$$

$$y(t) = (f * g)(t) + \alpha\cos(Kt) + \frac{\beta}{K}\sin(Kt)$$

④ Volterra Integral Equations: $x(t) = 2\cos(t) + 3\underbrace{\int_0^t \sin(t-y)\,x(y)\,dy}_{(\sin * x)(t)}$

$$X(s) = 2\mathcal{L}\{\cos(t)\} + 3\,\mathcal{L}\{\sin * x(t)\}$$

$$X(s) = 2\,\frac{s}{s^2+1} + 3\,\mathcal{L}\{\sin(t)\} \cdot \mathcal{L}\{x(t)\}$$

$$X(s) = \frac{2s}{s^2+1} + \frac{3}{s^2+1} X(s)$$

$$\left(1 - \frac{3}{s^2+1}\right) X(s) = \frac{2s}{s^2+1}$$

$$\frac{s^2 - 2}{s^2+1} X(s) = \frac{2s}{s^2+1}$$

$$X(s) = \frac{2s}{s^2 - 2} \Rightarrow x(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\}$$

$$\boxed{x(t) = 2\cosh(\sqrt{2}t)}$$

(5) $x(t) = t^3 + \underbrace{\int_0^t (t-y)\, x(y)\, dy}_{(f*x)(t),\ \text{where } f(t)=t \Rightarrow F(s)=\frac{1}{s^2}}$

$$X(s) = \mathcal{L}\{t^3\} + \mathcal{L}\{f*x(t)\}$$

$$X(s) = \frac{6}{s^4} + \frac{1}{s^2}X(s)$$

$$\left(1-\frac{1}{s^2}\right)X(s) = \frac{6}{s^4}$$

$$\frac{s^2-1}{s^2}X(s) = \frac{6}{s^4}$$

$$\boxed{X(s) = \frac{6}{s^2(s^2-1)}} \Rightarrow x(t) = -6t + 3e^t - 3e^{-t}$$
$$= -6t + 6\sinh(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\right\}$$

$$= \boxed{-t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}}$$

$$\frac{1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$1 = As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1)$$

$s=0$: $1 = -B$ $\quad B = -1$

$s=1$: $1 = 2C$ $\quad C = 1/2$

$s=-1$: $1 = -2D$ $\quad D = -1/2$

$s=2$: $1 = 6A + 3B + 12C + 4D$

$1 = 6A - 3 + 6 - 2$

$1 = 6A + 1$ $\quad A = 0$

OR

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{(s^2)-(s^2-1)}{s^2(s^2-1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-1} - \frac{1}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \sinh(t) - t$$

⑥ $y(t) = e^t + \underbrace{\int_0^t y(\tau)\,d\tau}_{}$

$(f*y)(t)$, where $f(t) = 1 \Rightarrow F(s) = \frac{1}{s}$

$$Y(s) = \mathcal{L}\{e^t\} + \mathcal{L}\{f*y\}$$
$$Y(s) = \frac{1}{s-1} + \frac{1}{s}Y(s)$$
$$\left(1 - \frac{1}{s}\right)Y(s) = \frac{1}{s-1}$$
$$\frac{s-1}{s}\,Y(s) = \frac{1}{s-1}$$
$$Y(s) = \frac{s}{(s-1)^2} \Rightarrow \boxed{y(t) = e^t(1+t)}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-1)+1}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2}\right\}$$
$$= e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-1}\right\}$$
$$= e^t + e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= e^t(1+t)$$

⑦ $y(t) = 1 + \int_0^t \overbrace{(\tau - t)}^{-(t-\tau)}\,y(\tau)\,d\tau$

$= 1 + (f*y)(t)$, where $f(t) = -t \Rightarrow F(s) = \frac{-1}{s^2}$

$$Y(s) = \mathcal{L}\{1\} + \mathcal{L}\{f*y\}$$
$$Y(s) = \frac{1}{s} + \frac{-1}{s^2}\cdot Y(s)$$
$$\left(1 + \frac{1}{s^2}\right)Y(s) = \frac{1}{s}$$
$$Y(s) = \frac{s}{s^2+1} \Rightarrow \boxed{y(t) = \cos(t)}$$

⑧ $y(t) = \int_0^t (\tau - t)\,y(\tau)\,d\tau = (f*y)(t)$, where $f(t) = -t$

$F(s) = \frac{-1}{s^2}$

$$\Rightarrow Y(s) = F(s)Y(s)$$
$$Y(s) = \frac{-1}{s^2}Y(s) \Rightarrow \frac{s^2+1}{s^2}Y(s) = 0 \Rightarrow Y(s) = 0 \Rightarrow \boxed{y(t) = 0}$$

9 RLC Circuit (Continued from class)

$$L = 0.1\,h \qquad R = 2\,\Omega \qquad C = 0.1\,f \qquad i(0) = 0$$
$$E(t) = 120t\,(1 - u_1(t))$$

Equation: $L \cdot s\,I(s) - \underbrace{L \cdot i(0)}_{=0} + R \cdot I(s) + \frac{1}{Cs} I(s) = \underbrace{\mathcal{L}\{E(t)\}}_{=}$

$$\mathcal{L}\{E(t)\} = 120\,\mathcal{L}\{t - t\,u_1(t)\}$$
$$= 120\left(\mathcal{L}\{t\} - \mathcal{L}\{[(t-1)+1]u_1(t)\}\right)$$
$$= 120\left(\frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\}\right)$$
$$= 120\left(\frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right)$$

$$\left((0.1)s + 2 + \frac{1}{(0.1)s}\right) I(s)$$
$$= 120\,\frac{1 - e^{-s} - se^{-s}}{s^2}$$

$$\frac{s}{10} + 2 + \frac{10}{s} = \frac{s^2+100}{10s} + 2$$
$$= \frac{s^2 + 20s + 100}{10s}$$
$$= \frac{(s+10)^2}{10s}$$

$$\frac{(s+10)^2}{10s} I(s) = 120\,\frac{1 - e^{-s} - se^{-s}}{s^2} \quad \Big| \cdot \frac{10s}{(s+10)^2}$$

$$I(s) = 1200\,\frac{(1 - e^{-s} - se^{-s})}{s(s+10)^2}$$

$$\Rightarrow i(t) = 1200\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2} - \frac{e^{-s}}{s(s+10)^2} - \frac{e^{-s}}{(s+10)^2}\right\}$$

$$\frac{1}{s(s+10)^2} = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{(s+10)^2}$$
$$1 = A(s+10)^2 + Bs(s+10) + Cs$$

$s=0$: $1 = 100A$ $\quad A = 1/100$

$s=-10$: $1 = -10C$ $\quad C = -1/10$

$s=-9$: $1 = A - 9B - 9C$
$$1 = \frac{1}{100} - 9B + \frac{9}{10}$$
$$9B = \frac{91}{100} - 1 = \frac{-9}{100}$$
$$B = -1/100$$

$$\frac{1}{s^2}\Big|_{s \to s+10}$$
$$\downarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}\right\} =$$
$$= \frac{1}{100}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{100}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{(s+10)^2}\right\}$$
$$= \frac{1}{100} - \frac{1}{100}e^{-10t} - \frac{1}{10}e^{-10t} \cdot t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+10)^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+10)^2}\right\}\Big|_{t \to t-1} u_1(t)$$
$$= \left(\frac{1}{100} - \frac{1}{100}e^{-10(t-1)} - \frac{1}{10}e^{-10(t-1)}(t-1)\right)u_1(t)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+10)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+10)^2}\right\}\Big|_{t\to t-1} u_1(t)$$

$$= \left(e^{-10t}\cdot t\right)\Big|_{t\to t-1} u_1(t)$$

$$= e^{-10(t-1)}(t-1)\, u_1(t)$$

$$\Rightarrow i(t) = 1200\Big(\frac{1}{100} - \frac{1}{100}e^{-10t} - \frac{1}{10}e^{-10t}\cdot t - \frac{1}{100}u_1(t) + \frac{1}{100}e^{-10(t-1)}u_1(t) + \frac{1}{10}e^{-10(t-1)}(t-1)u_1(t) - e^{-10(t-1)}(t-1)u_1(t)\Big)$$

$$= 1200\Big(\frac{1}{100} - \frac{1}{100}u_1(t) - \frac{1}{100}e^{-10t} - \frac{1}{10}e^{-10t}\cdot t + \frac{1}{100}e^{-10(t-1)}u_1(t) - \frac{9}{10}e^{-10(t-1)}(t-1)u_1(t)\Big)$$