Math 308 - Differential Equations **Homework 12**
Section 501
Texas A&M University, Spring 2022.

Laplace Transform:
Convolutions and Laplace Transform

1. Use convolutions to find

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\},$$

where $k \neq 0$ is a constant.

2. Find

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{(s-1)^2+1}\right\},$$

where $G(s)$ is the Laplace transform of a piecewise continuous function $g(t)$ of exponential order. Specifically, your answer should be $(f \star g)(t)$, where you specify the function $f(t)$ explicitly (g is any appropriate function).

3. Find the solution to the initial value problem:

$$y'' + k^2 y = g(t);\;\; y(0) = \alpha,\; y'(0) = \beta,$$

where $k \neq 0$, α and β are constants, and $g(t)$ is a piecewise continuous function of exponential order. Your answer will contain a convolution $f \star g(t)$, you must only specify the function $f(t)$ explicitly.

Solve the following equations – all examples of **Volterra integral equations**:

4. $x(t) = 2\cos t + 3\int_0^t \sin(t-y)x(y)\,dy.$

5. $x(t) = t^3 + \int_0^t (t-y)x(y)\,dy.$

6. $y(t) = e^t + \int_0^t y(\tau)\,d\tau.$

7. $y(t) = 1 + \int_0^t (\tau - t)y(\tau)\,d\tau.$

8. $y(t) = \int_0^t (\tau - t)y(\tau)\,d\tau.$

9. Kirchhoff's second law in a series RLC circuit (consisting of a resistor, an inductor, and a capacitor) is:

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t),$$

where $E(t)$ is the voltage. Find the current $i(t)$ in such an RLC circuit, where

$$L = 0.1h;\; R = 2\Omega;\; C = 0.1f;\;\; i(0) = 0,$$

and the voltage function is given by

$$E(t) = 120t(1 - u_1(t)).$$

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\ s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\ s > 0$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a};\ s > a$

$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2};\ s > 0$

$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2};\ s > 0$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\ s > 0;\ a > 0.$

$\mathscr{L}\{\sinh(at)\} = \frac{a}{s^2-a^2};\ s > |a|$

$\mathscr{L}\{\cosh(at)\} = \frac{s}{s^2-a^2};\ s > |a|$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0};\ (t_0 > 0).$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0);\ (t_0 > 0).$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) = \mathscr{L}\{f(t)\}|_{s\to s-a}$

$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\left\{F(s)|_{s\to s-a}\right\} = e^{at}\mathscr{L}^{-1}\{F(s)\} = e^{at}f(t)$

Translation Theorem II:

$\mathscr{L}\{f(t-a)u_a(t)\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$

$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}u_a(t)$

Derivatives of Laplace Transforms:

$\mathscr{L}\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$

$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\ \vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$

$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$

Dirac Delta Function:

$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$

$(f\star\delta)(t) = f(t)$