

**Laplace Transform:
Dirac Delta Function and Laplace Transform**

Solve the differential equations:

1. $y' - 3y = \delta(t - 2); y(0) = 0.$
2. $y'' + y = \delta(t - 2\pi); y(0) = 0, y'(0) = 1.$
3. $y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right); y(0) = y'(0) = 0.$
4. $y'' + 2y' = \delta(t - 1); y(0) = 0, y'(0) = 1.$
5. $y'' + 4y' + 5y = \delta(t - 2\pi); y(0) = y'(0) = 0.$
6. $y'' + 4y' + 13y = \delta(t - \pi) + \delta(t - 3\pi); y(0) = 1, y'(0) = 0.$

Laplace transforms of some basic functions

$$\begin{aligned} \mathcal{L}\{1\} &= \frac{1}{s}; \quad s > 0 & \mathcal{L}\{\sin(at)\} &= \frac{a}{s^2 + a^2}; \quad s > 0 & \mathcal{L}\{\sinh(at)\} &= \frac{a}{s^2 - a^2}; \quad s > |a| \\ \mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}; \quad s > 0 & \mathcal{L}\{\cos(at)\} &= \frac{s}{s^2 + a^2}; \quad s > 0 & \mathcal{L}\{\cosh(at)\} &= \frac{s}{s^2 - a^2}; \quad s > |a| \\ \mathcal{L}\{e^{at}\} &= \frac{1}{s - a}; \quad s > a & \mathcal{L}\{u_a(t)\} &= \frac{e^{-as}}{s}; \quad s > 0; \quad a > 0. & \mathcal{L}\{\delta(t - t_0)\} &= e^{-st_0}; \quad (t_0 > 0). \end{aligned}$$

Inverse Laplace transforms of some basic functions

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 & \mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} &= \frac{1}{a} \sin(at) & \mathcal{L}^{-1}\left\{\frac{1}{s^2 - a^2}\right\} &= \frac{1}{a} \sinh(at) \\ \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} &= \frac{1}{(n-1)!} t^{n-1} & \mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} &= \cos(at) & \mathcal{L}^{-1}\left\{\frac{s}{s^2 - a^2}\right\} &= \cosh(at) \\ \mathcal{L}^{-1}\left\{\frac{1}{s - a}\right\} &= e^{at} & \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} &= u_a(t) & \mathcal{L}^{-1}\{e^{-st_0}\} &= \delta(t - t_0); \quad (t_0 > 0). \end{aligned}$$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\begin{aligned} \mathcal{L}\{e^{at}f(t)\} &= F(s - a) = \mathcal{L}\{f(t)\}|_{s \rightarrow s-a} \\ \mathcal{L}^{-1}\{F(s - a)\} &= \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s-a}\} = e^{at} \mathcal{L}^{-1}\{F(s)\} = e^{at} f(t) \end{aligned}$$

Translation Theorem II:

$$\begin{aligned} \mathcal{L}\{f(t - a)u_a(t)\} &= e^{-as}F(s) = e^{-as} \mathcal{L}\{f(t)\} \\ \mathcal{L}^{-1}\{e^{-as}F(s)\} &= f(t - a)u_a(t) = \mathcal{L}^{-1}\{F(s)\}|_{t \rightarrow t-a} u_a(t) \end{aligned}$$

Derivatives of Laplace Transforms:

$$\begin{aligned} \mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n}{ds^n} F(s) \\ \mathcal{L}^{-1}\{F^{(n)}(s)\} &= (-1)^n t^n f(t) \end{aligned}$$

Laplace Transform of Periodic Functions:

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T :

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned} \mathcal{L}\{y'\} &= sY(s) - y(0) \\ \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) \\ \mathcal{L}\{y'''\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ &\vdots \\ \mathcal{L}\{y^{(n)}(t)\} &= s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - sy^{(n-2)}(0) - y^{(n-1)}(0) \end{aligned}$$

Convolution Theorem:

$$\begin{aligned} \mathcal{L}\{f(t) \star g(t)\} &= F(s)G(s) = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} \\ \mathcal{L}^{-1}\{F(s)G(s)\} &= f(t) \star g(t) \end{aligned}$$

Dirac Delta Function:

$$\begin{aligned} \int_0^\infty f(t) \delta(t - t_0) dt &= f(t_0) \\ (f \star \delta)(t) &= f(t) \end{aligned}$$