Homework 14 Solutions

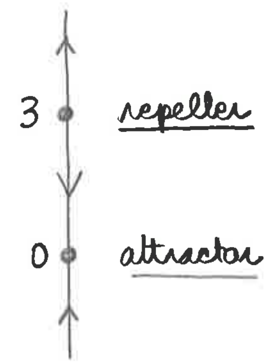
1 Attractors & Repellers

(a) $y' = y^2 - 3y$

$f(y) = y^2 - 3y = y(y-3)$

$f(y) = 0 \Rightarrow \boxed{y=0;\ y=3}$.

y			0		3			
$y(y-3)$	+	+	0	−	0	+	+	+



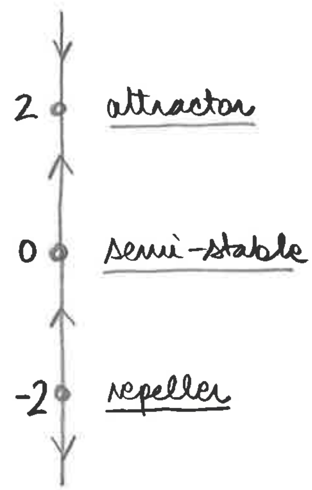
(b) $y' = y^2(4-y^2)$

$f(y) = y^2(4-y^2)$

$f(y) = 0 \Rightarrow \boxed{y=0;\ y=2;\ y=-2}$

equilibrium solutions

y		−2		0		2	
$4-y^2$	−	0	+		+	0	−
y^2	+		+	0	+		+
$f(y)$	−	0	+	0	+	0	−

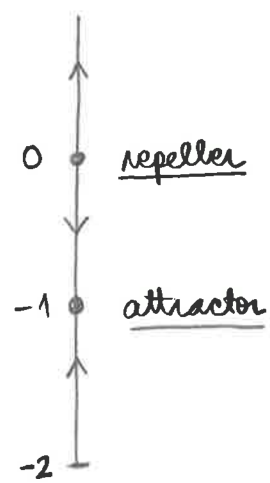


(c) $y' = y \ln(y+2)$

$f(y) = y\ln(y+2) = 0 \Rightarrow \boxed{y=0,\ y=-1}$

equilibrium solns.

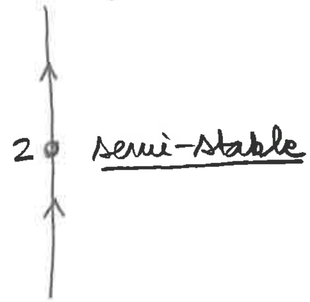
	−2		−1		0		∞
y	−	−	−		0	+	+
$\ln(y+2)$	−	0	+	+	+		+
$f(y)$	+	0	−	0		+	+



(d) $y' = (y-2)^4$

$f(y) = (y-2)^4 = 0 \Rightarrow \boxed{y=2}$ equil. sol.

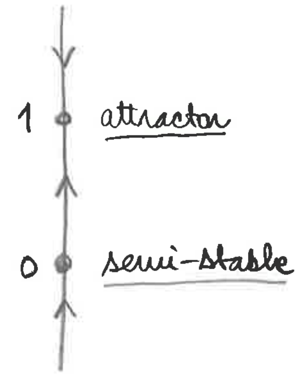
$(y-2)^4 \geq 0$ for all y



(e) $y' = y^2 - y^3$

$f(y) = y^2(1-y) \Rightarrow \boxed{y=0;\ y=1}$ equil. sol.

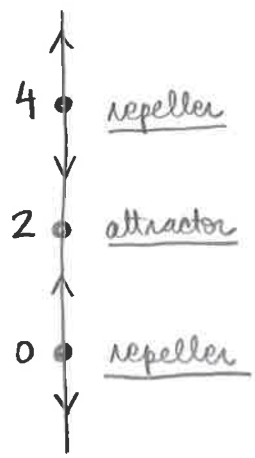
		0		1	
$1-y$	+	+	+	0	−
y^2	+	0	+	+	+
$f(y)$	+	0	+	0	−



(f) $y' = y(2-y)(4-y)$

$f(y) = y(2-y)(4-y) \Rightarrow \boxed{y=0;\ y=2;\ y=4}$ equil. sol.

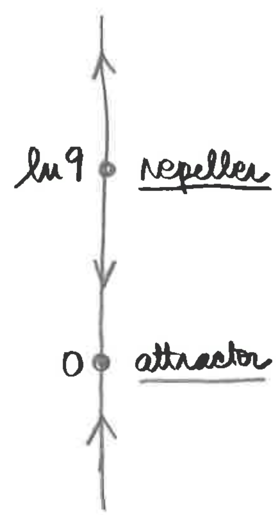
		0		2		4	
y	−	0	+	+	+	+	+
$2-y$	+	+	+	0	−	−	−
$4-y$	+	+	+	+	+	0	−
$f(y)$	−	0	+	0	−	0	+



(g) $y' = \dfrac{ye^y - 9y}{e^y}$

$f(y) = \dfrac{y(e^y-9)}{e^y} = 0 \Rightarrow \boxed{y=0;\ y=\ln 9}$ equil. sol.

		0		ln9	
y	−	0	+	+	+
e^y-9	−	−	−	0	+
$f(y)$	+	0	−	0	+



② Logistic Equation:

$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{1}{K}y\right)y} \quad \begin{array}{l} r>0 \\ K>0 \end{array}$$

Solution subject to $y(0) = y_0$:

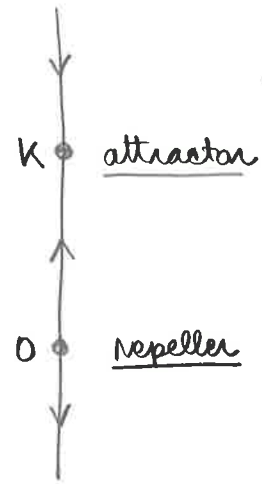
$$y(t) = \frac{Ky_0}{y_0 + (K - y_0)e^{-rt}} \;; t > 0$$

(a) Equilibrium Solutions:

$$y' = r\left(1 - \frac{1}{K}y\right)y$$

$$y' = 0 \Rightarrow \boxed{y = 0 ;\; y = K}$$

		0		K	
y	$-$	0	$+$	$+$	$+$
$1 - \frac{1}{K}y$	$+$	$+$	$+$	0	$-$
y'	$-$	0	$+$	0	$-$



(b) Suppose $0 < y_0 < K$. Show that $y(t)$ is increasing on $(0, \infty)$, and

$$0 < y(t) \leq K, \quad \forall t > 0.$$

Compute $y'(t)$:

$$y'(t) = -\frac{Ky_0}{(y_0 + (K - y_0)e^{-rt})^2}\left((K - y_0)\cdot(-r)e^{-rt}\right)$$

$$= \frac{r y_0 K (K - y_0)\, e^{-rt}}{(y_0 + (K - y_0)e^{-rt})^2} > 0 \quad \Rightarrow y(t) \text{ is increasing}$$

It is obvious that $y(t) > 0$ for all t. To prove the other inequality:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{Ky_0}{y_0 + (K - y_0)\underbrace{e^{-rt}}_{\xrightarrow{t\to\infty} 0}} = \frac{Ky_0}{y_0} = K$$

So y is increasing and $\lim_{t\to\infty} y(t) = K$, so $y(t) \leq K$ for all $t > 0$.

$\Rightarrow$ If the initial population is smaller than the carrying capacity, the population increases over time and approaches the carrying capacity over time.

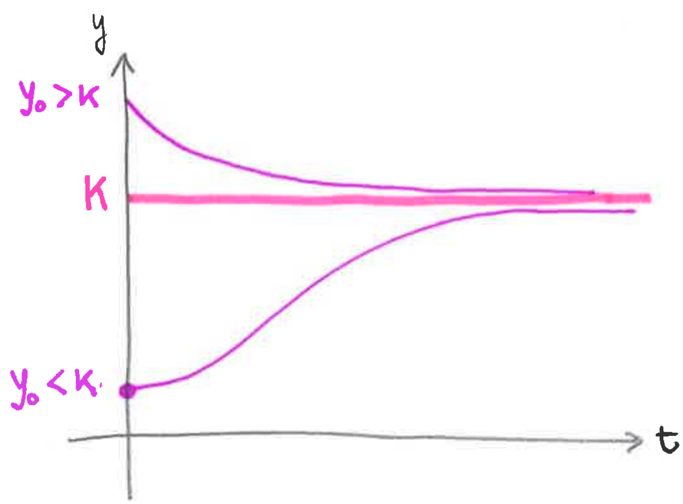
(c). Suppose $y_0 > K$. Then

$$y'(t) = \frac{r y_0 K (K - y_0) e^{-rt}}{(y_0 + (K - y_0) e^{-rt})^2} < 0 \Rightarrow y(t) \text{ is \underline{decreasing}}$$

The limit as $t \to \infty$ remains $\lim_{t\to\infty} y(t) = K$, so because y is decreasing

$$y(t) \geq K \text{ for all } t.$$

$\Rightarrow$ If the initial population exceeds the carrying capacity, the population decreases, stabilizing at the carrying capacity.



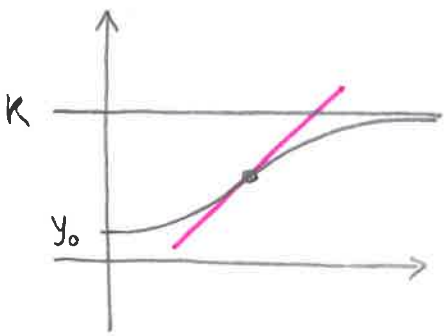
0 only if y = 0 or y = K,
neither of which are possib
on this interval.

(d) $y' = r\left(1 - \frac{1}{K}y\right)y$

$$y'' = r\left(-\frac{1}{K}y + \left(1 - \frac{1}{K}y\right)\right) = r\left(1 - \frac{2}{K}y\right)y'$$

(Recall that an inflection point of a function $f(x)$ is a point where the second derivative $f''(x)$ changes sign, and f changes concavity).

$y'' = 0 \Rightarrow \boxed{y = \frac{K}{2}} \Rightarrow$ The inflection point occurs when the population reaches $K/2$.



$\Rightarrow$ This can only happen if $y_0 < K$ (because if $y_0 > K$, then $y(t) \geq K$ for all t).

But y'' is the derivative of y' (the rate of growth), so the sign of y'' tells us the monotonicity of y'

			$K/2$			
y''	+	+	0	−	−	−
y'	$\nearrow$		max		$\searrow$	

$\Rightarrow$ The inflection point is where the rate of growth is maxim

$\Rightarrow$ The population initially increases steadily from y_0.
Once the population reaches half the carrying capacity, it begins to grow slower and slower, eventually tapering off around K.

3. $y_0 = 3$ mg
$K = 100$ mg
$r = 0.2$/hour

$$\frac{dy}{dt} = 0.2\, y\left(1 - \frac{1}{100}y\right)$$

$$\boxed{y(t) = \frac{300}{3 + 97e^{-0.2t}}}$$

(a) $y(t) = 20$ mg

$$\Rightarrow \frac{300}{3+97e^{-0.2t}} = 20 \Rightarrow 3+97e^{-0.2t} = 15 \Rightarrow e^{-0.2t} = \frac{12}{97}$$

$$\Rightarrow -0.2t = \ln(12/97)$$

$$\Rightarrow \boxed{t = \frac{\ln(97/12)}{0.2}} \simeq 10.45 \text{ hours}$$

(b) $y(t) = 200$ mg

The long way : $\frac{300}{3+97e^{-0.2t}} = 200 \Rightarrow 3+97e^{-0.2t} = \frac{3}{2}$

$$\Rightarrow 97e^{-0.2t} = -\frac{3}{2}$$

No solutions : $97e^{-0.2t} > 0 \ \forall t$

The short way : Never, because 200 mg exceeds the carrying capacity.