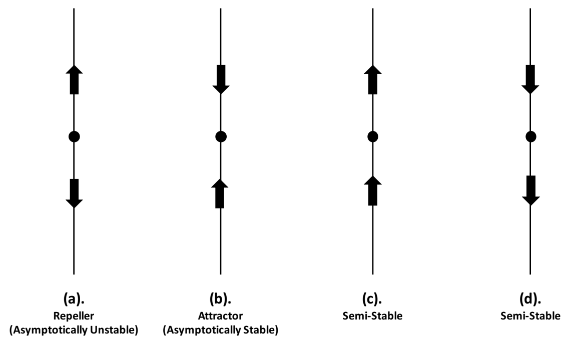
Math 308 - Differential Equations **Homework 14**
Texas A&M, Spring 2022; Section 501

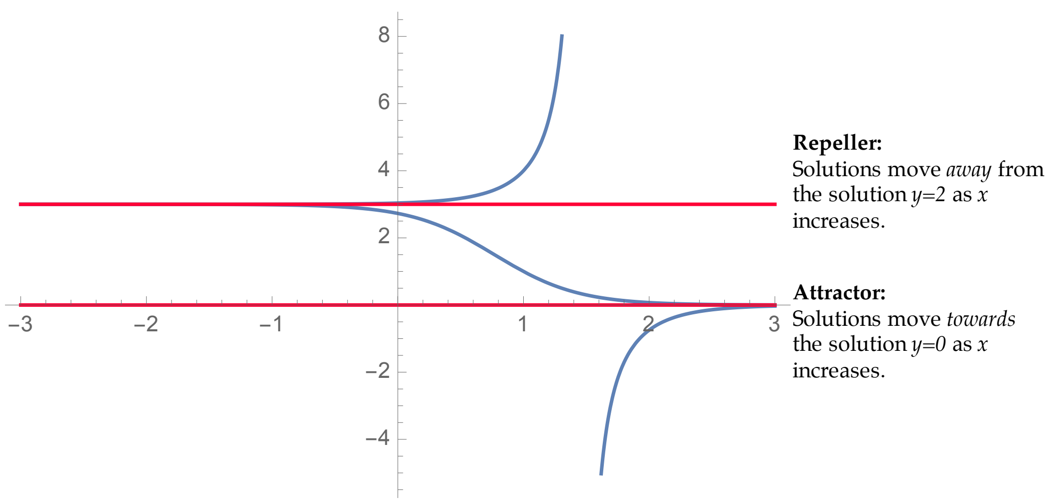
Autonomous Equations; Population Models

1. Attractors and Repellers. The figure below illustrates the four possible behaviors of critical points c of an autonomous ODE, in terms of phase portraits.

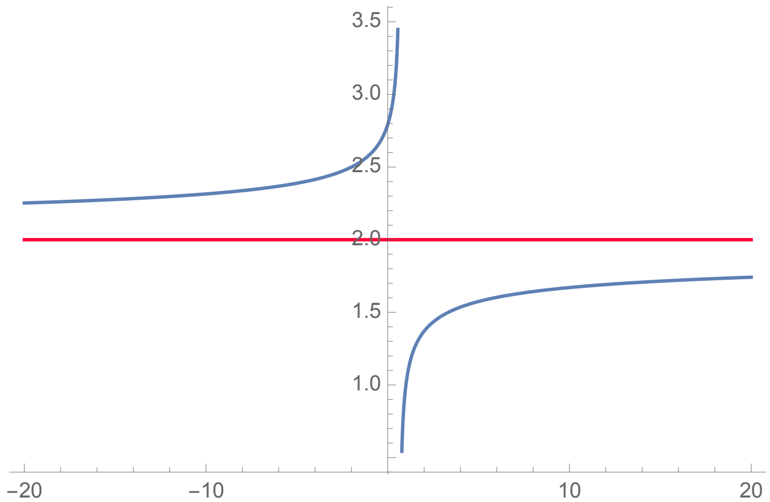


(1). The arrows are both pointing *away* from the critical point c, as in Figure (a). This means that any solution passing close enough to c moves *away* from c as $x \to \infty$. In this case, the critical point c is said to be **asymptotically unstable**, or a **repeller**.

(2). The arrows both point *towards* the critical point, asin Figure (b), which means that any solutions passing close enough to c have the asymptotic behavior $\lim_{x\to\infty} y(x) = c$. In this case c is said to be **asymptotically stable**, or an **attractor**.



(3). The situations in Figures (c) and (d), where one arrow points towards and one away from c, are neither attractors nor repellers. However, such points display properties of both, since they "attract" solutions on one side, but "repel" solutions on the other side. Then c is said to be **semi-stable**.



Semi-Stable:
Solutions *above* the line *y*=2 move *away* from it, while solutions *below* this line move *towards* it as *x* increases.

For each of the autonomous equations below:

- Find the critical points and equilibrium solutions.
- Draw the phase portrait.
- Classify each critical point as an attractor, repeller, or semi-stable.

a. $y' = y^2 - 3y$.

b. $y' = y^2(4 - y^2)$.

c. $y' = y\ln(y + 2)$.

d. $y' = (y - 2)^4$.

e. $y' = y^2 - y^3$.

f. $y' = y(2 - y)(4 - y)$.

g. $y' = \dfrac{ye^y - 9y}{e^y}$.

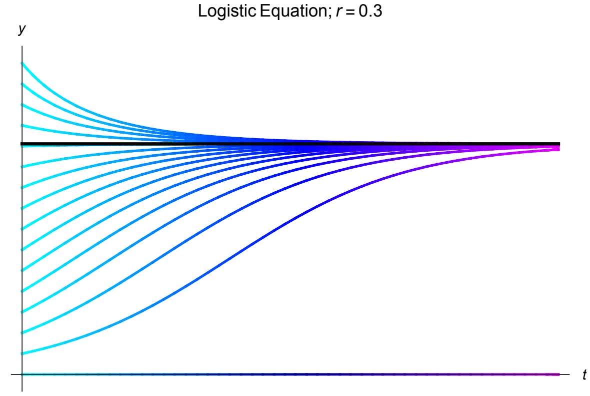
2. The Logistic Equation: In what follows, consider the logistic equation:

$$\frac{dy}{dt} = r\left(1 - \frac{1}{K}y\right)y, \qquad (\star)$$

where $r > 0$ is a fixed growth constant and $K > 0$ is the fixed **carrying capacity** (the maximum population the environment can sustain). Recall that we are usually only interested in solving this equation for $t > 0$, since t represents time. Solving this subject to an initial condition $y(0) = y_0$ yields the solution:

$$y(t) = \frac{Ky_0}{y_0 + (K - y_0)e^{-rt}}.$$

Below is a typical graph of this solution, with varying initial conditions.



(a). Find the equilibrium solutions of $(\star)$, draw its phase portrait, and classify the critical points as attractors or repellers.

(b). Suppose that $0 < y_0 < K$. Show that the solution $y(t)$ is then *increasing* on $t \in (0, \infty)$, with

$$0 \leq y(t) \leq K, \text{ for all } t > 0.$$

Interpret the meaning of this for the population model.

(c). Suppose that $y_0 > K$. Show that the solution $y(t)$ is then *decreasing* on $t \in (0, \infty)$, with

$$y(t) \geq K, \text{ for all } t > 0.$$

Interpret the meaning of this for the population model.

(d). Suppose $y(t)$ is a solution of $(\star)$. Where is the inflection point of the solution? (In terms of y, not t. In other words, if the solution has an inflection point $(t_0, y(t_0))$ for some $t_0 > 0$, find the population $y(t_0)$ at this time - not interested in the value of t_0.) Note that you can do this directly from $(\star)$, you do not need to work with the explicit solution.

What is the significance of this point for the *rate of growth* of the population? Interpret the meaning of this inflection point for the population model.

3. A run-of-the-mill population problem: Suppose a population of bacteria follows the logistic growth model. Suppose further that the initial population is 3mg of bacteria, the carrying capacity is 100mg, and the growth parameter is $r = 0.2$/hour.

a). At what time does the population reach 20mg?

b). At what time does the population reach 200mg?