

Solving Systems of ODEs with the Laplace Transform

$$\textcircled{1} \begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = 2x \end{cases}$$

$$x(0) = 0; y(0) = 1.$$

$$\begin{cases} sX(s) = -X(s) + Y(s) \\ sY(s) - 1 = 2X(s) \end{cases} \quad \begin{cases} (s+1)X(s) - Y(s) = 0 \\ -2X(s) + sY(s) = 1 \end{cases}$$

$$-2Y(s) + s(s+1)Y(s) = s+1$$

$$Y(s)(s^2+s-2) = s+1 \Rightarrow Y(s) = \frac{s+1}{(s-1)(s+2)}$$

$$s(s+1)X(s) - 2X(s) = 1$$

$$X(s)(s^2+s-2) = 1 \Rightarrow X(s) = \frac{1}{(s-1)(s+2)}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{s+1}{(s-1)(s+2)} \right\} \\ = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$$

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+2)} \right\} \\ = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

$$x(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

$$y(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$$

$$\frac{s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$\frac{s+1}{s+2} \Big|_{s=1} = A$$

$$\frac{s+1}{s-1} \Big|_{s=-2} = B$$

$$A = \frac{2}{3}$$

$$B = \frac{1}{3}$$

$$\frac{1}{(s-1)(s+2)} = \frac{(s+2) - (s-1)}{(s-1)(s+2)} \cdot \frac{1}{3} = \frac{1/3}{s-1} - \frac{1/3}{s+2}$$

$$\textcircled{2} \begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 5x - y \end{cases}$$

$$x(0) = -1; y(0) = 2$$

$$\begin{cases} sX(s) + 1 = X(s) - 2Y(s) \\ sY(s) - 2 = 5X(s) - Y(s) \end{cases}$$

$$\begin{cases} (s-1)X(s) + 2Y(s) = -1 & \left| \begin{matrix} (\cdot 5) \\ + (s+1) \end{matrix} \right. \\ -5X(s) + (s+1)Y(s) = 2 & \left| \begin{matrix} (\cdot (-1)) \\ -2 \end{matrix} \right. \end{cases}$$

$$10Y(s) + (s^2-1)Y(s) = -5 + 2(s-1)$$

$$(s^2+9)Y(s) = 2s-7$$

$$Y(s) = \frac{2s-7}{s^2+9}$$

$$(s^2-1)X(s) + 10X(s) = -s-1-4$$

$$(s^2+9)X(s) = -s-5$$

$$X(s) = -\frac{s+5}{s^2+9}$$

$$x(t) = -\cos(3t) - \frac{5}{3}\sin(3t)$$

$$y(t) = 2\cos(3t) - \frac{7}{3}\sin(3t)$$

$$\textcircled{3} \begin{cases} 2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2 \end{cases}$$

$$x(0) = 0; y(0) = 0$$

$$\begin{cases} 2sX(s) + sY(s) - 2X(s) = \frac{1}{s} \\ sX(s) + sY(s) - 3X(s) - 3Y(s) = \frac{2}{s} \end{cases}$$

$$\begin{cases} (2s-2)X(s) + sY(s) = \frac{1}{s} \\ (s-3)X(s) + (s-3)Y(s) = \frac{2}{s} \end{cases}$$

$$(2s-2)X(s) + \frac{2}{s-3} - 5X(s) = \frac{1}{s}$$

$$(s-2)X(s) = \frac{1}{s} - \frac{2}{s-3}$$

$$X(s) = \frac{1}{s(s-2)} - \frac{2}{(s-2)(s-3)}$$

$$= \frac{5}{2} \frac{1}{s-2} - \frac{2}{s-3} - \frac{1}{2} \frac{1}{s}$$

$$Y(s) = \frac{2}{s(s-3)} - X(s)$$

$$\Rightarrow Y(s) = \frac{2}{s(s-3)} - X(s)$$

$$= \frac{8}{3} \frac{1}{s-3} - \frac{5}{2} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s}$$

$$x(t) = \frac{5}{2}e^{2t} - 2e^{3t} - \frac{1}{2}$$

$$y(t) = \frac{8}{3}e^{3t} - \frac{5}{2}e^{2t} - \frac{1}{6}$$

$$\textcircled{4} \begin{cases} \frac{d^2x}{dt^2} + x - y = 0 \\ \frac{d^2y}{dt^2} + y - x = 0 \end{cases}$$

$$\begin{aligned} x(0) &= 0 & y(0) &= 0 \\ x'(0) &= -2 & y'(0) &= 1 \end{aligned}$$

$$\begin{cases} s^2x(s) + 2 + x(s) - y(s) = 0 \\ s^2y(s) - 1 + y(s) - x(s) = 0 \end{cases}$$

$$\begin{cases} (s^2+1)x(s) - y(s) = -2 & \times 1 & \times (s^2+1) \\ -x(s) + (s^2+1)y(s) = +1 & \times (s^2+1) & \times 1 \end{cases}$$

$$-y(s) + (s^2+1)^2 y(s) = -2 + s^2 + 1$$

$$((s^2+1)^2 - 1)y(s) = s^2 - 1$$

$$\begin{aligned} y(s) &= \frac{s^2 - 1}{s^2(s^2 + 2)} = \frac{1}{s^2 + 2} - \frac{1}{s^2(s^2 + 2)} \\ &= \frac{1}{s^2 + 2} - \frac{1}{2} \frac{(s^2 + 2) - s^2}{s^2(s^2 + 2)} \\ &= \frac{1}{s^2 + 2} - \frac{1/2}{s^2} + \frac{1/2}{s^2 + 2} \\ &= \frac{3/2}{s^2 + 2} - \frac{1/2}{s^2} \end{aligned}$$

$$\Rightarrow y(t) = \frac{3}{2\sqrt{2}} \sin(\sqrt{2}t) - \frac{1}{2}t$$

$$(s^2+1)^2 x(s) - x(s) = -2(s^2+1) + 1$$

$$s^2(s^2+2)x(s) = -2s^2 - 1$$

$$\begin{aligned} x(s) &= -\frac{2s^2 + 1}{s^2(s^2 + 2)} \\ &= -\frac{2}{s^2 + 2} - \frac{1}{s^2(s^2 + 2)} \\ &= -\frac{2}{s^2 + 2} - \frac{1/2}{s^2} + \frac{1/2}{s^2 + 2} \\ &= \frac{-3/2}{s^2 + 2} - \frac{1/2}{s^2} \end{aligned}$$

$$\Rightarrow x(t) = -\frac{3}{2\sqrt{2}} \sin(\sqrt{2}t) - \frac{1}{2}t$$

$$\textcircled{5} \begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2 \\ \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = 4t \end{cases}$$

$$\begin{aligned} x(0) &= 8; & y(0) &= 0 \\ x'(0) &= 0; & y'(0) &= 0. \end{aligned}$$

$$\begin{cases} s^2 X(s) - 8s + s^2 Y(s) = \frac{2}{s^3} \\ s^2 X(s) - 8s - s^2 Y(s) = \frac{4}{s^2} \end{cases}$$

$$\begin{cases} s^2 X(s) + s^2 Y(s) = \frac{2}{s^3} + 8s \\ s^2 X(s) - s^2 Y(s) = \frac{4}{s^2} + 8s \end{cases}$$

$$\begin{cases} X(s) + Y(s) = \frac{2}{s^5} + \frac{8}{s} \\ X(s) - Y(s) = \frac{4}{s^4} + \frac{8}{s} \end{cases}$$

$$2Y(s) = \frac{2}{s^5} - \frac{4}{s^4}$$

$$Y(s) = \frac{1}{s^5} - \frac{2}{s^4}$$

$$y(t) = \frac{1}{24}t^4 - \frac{1}{3}t^3$$

$$2X(s) = \frac{2}{s^5} + \frac{4}{s^4} + \frac{16}{s}$$

$$X(s) = \frac{1}{s^5} + \frac{2}{s^4} + \frac{8}{s}$$

$$\Rightarrow x(t) = \frac{1}{24}t^4 + \frac{1}{3}t^3 + 8$$

$$\textcircled{6} \begin{cases} \frac{d^2x}{dt^2} + 3\frac{dy}{dt} + 3y = 0 \\ \frac{d^2x}{dt^2} + 3y = te^{-t} \end{cases}$$

$$\begin{aligned} x(0) &= 0; & y(0) &= 0 \\ x'(0) &= 2; \end{aligned}$$

$$\begin{cases} s^2 X(s) - 2 + 3sY(s) + 3Y(s) = 0 \\ s^2 X(s) - 2 + 3Y(s) = \frac{1}{(s+1)^2} \end{cases}$$

$$\begin{cases} s^2 X(s) + 3(s+1)Y(s) = 2 \\ s^2 X(s) + 3Y(s) = 2 + \frac{1}{(s+1)^2} \end{cases}$$

⊖

$$3sY(s) = -\frac{1}{(s+1)^2} \Rightarrow Y(s) = \frac{-1}{3s(s+1)^2}$$

$$\begin{aligned} X(s) &= \frac{2}{s^2} + \frac{3(s+1)}{s^2} \frac{1}{3s(s+1)^2} \\ &= \frac{2}{s^2} + \frac{1}{s^3(s+1)} \end{aligned}$$

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A=1$$

$$C=-1$$

$$B=-1$$

$$\frac{B}{s+1} = \frac{1}{s(s+1)^2} - \frac{1}{s} + \frac{1}{(s+1)^2}$$

$$= \frac{1 - (s+1)^2 + s}{s(s+1)^2} = \frac{1 - s^2 - 2s - 1 + s}{s(s+1)^2} = -\frac{s+1}{(s+1)^2} = -\frac{1}{s+1}$$

$$y(t) = -\frac{1}{3} \left(1 - e^{-t} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \right)$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2} \Big|_{s \rightarrow s+1} \right\} = te^{-t}$$

$$y(t) = -\frac{1}{3} + \frac{1}{3}e^{-t} + \frac{1}{3}te^{-t}$$

$$\frac{1}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$$

$$D=-1$$

$$C=1$$

$$A=1 \quad B=-1$$

$$\begin{aligned} \frac{A}{s} + \frac{B}{s^2} &= \frac{1}{s^3(s+1)} - \frac{1}{s^3} + \frac{1}{s+1} = \frac{1 - s - 1 + s^3}{s^3(s+1)} = \frac{s(s-1)(s+1)}{s^3(s+1)} \\ &= \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2} \end{aligned}$$

$$\Rightarrow x(t) = 2t + 1 - t + \frac{1}{2}t^2 - e^{-t}$$

$$x(t) = 1 + t + \frac{1}{2}t^2 - e^{-t}$$

Another way to solve #6 :

$$\begin{cases} \frac{d^2x}{dt^2} + 3 \frac{dy}{dt} + 3y = 0 & (1) \end{cases}$$

$$\begin{cases} \frac{d^2x}{dt^2} + 3y = te^{-t} & (2) \end{cases}$$

$$x(0) = 0 ; y(0) = 0$$

$$x'(0) = 2 ;$$

$$(1) - (2) \Rightarrow 3 \frac{dy}{dt} = -te^{-t} \Rightarrow \frac{dy}{dt} = -\frac{1}{3}te^{-t}$$

$$\Rightarrow y(t) = -\frac{1}{3} \int te^{-t} dt$$

$$= \frac{1}{3} \int t(e^{-t})' dt$$

$$= \frac{1}{3} (te^{-t} - \int e^{-t} dt)$$

$$= \frac{1}{3} (te^{-t} + e^{-t} + c)$$

$$y(0) = 0 \Rightarrow 0 = \frac{1}{3}(1+c) \Rightarrow c = -1 \Rightarrow \boxed{y(t) = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}}$$

$$\text{Sub in (2)} \Rightarrow \frac{d^2x}{dt^2} = te^{-t} - 3y \Rightarrow \frac{d^2x}{dt^2} = -e^{-t} + 1$$

$$\Rightarrow \frac{dx}{dt} = e^{-t} + t + c_1$$

$$\Rightarrow x(t) = -e^{-t} + \frac{t^2}{2} + c_1t + c_2$$

$$x'(0) = 2 \Rightarrow 2 = 1 + c_1 \Rightarrow c_1 = 1$$

$$x(0) = 0 \Rightarrow 0 = -1 + c_2 \Rightarrow c_2 = 1$$

$$\Rightarrow \boxed{x(t) = -e^{-t} + \frac{t^2}{2} + t + 1}$$