Math 308 - Differential Equations **Homework 15**
Section 501
Texas A&M, Spring 2022

Solving Systems of Differential Equations with the Laplace Transform

Use the Laplace transform method to solve the following systems of differential equations:

1.
$$\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = 2x \end{cases}$$
$x(0) = 0;\ y(0) = 1.$

2.
$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 5x - y \end{cases}$$
$x(0) = -1;\ y(0) = 2.$

3.
$$\begin{cases} 2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2 \end{cases}$$
$x(0) = 0;\ y(0) = 0.$

4.
$$\begin{cases} \frac{d^2x}{dt^2} + x - y = 0 \\ \frac{d^2y}{dt^2} + y - x = 0 \end{cases}$$
$x(0) = 0;\ x'(0) = -2;$
$y(0) = 0; y'(0) = 1.$

5.
$$\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2 \\ \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = 4t \end{cases}$$
$x(0) = 8;\ x'(0) = 0;$
$y(0) = 0;\ y'(0) = 0.$

6.
$$\begin{cases} \frac{d^2x}{dt^2} + 3\frac{dy}{dt} + 3y = 0 \\ \frac{d^2x}{dt^2} + 3y = te^{-t} \end{cases}$$
$x(0) = 0;\ x'(0) = 2;$
$y(0) = 0.$

Laplace transforms of some basic functions

$\mathscr{L}\{1\} = \frac{1}{s};\; s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}};\; s > 0$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a};\; s > a$

$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2};\; s > 0$

$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2};\; s > 0$

$\mathscr{L}\{\sinh(at)\} = \frac{a}{s^2-a^2};\; s > |a|$

$\mathscr{L}\{\cosh(at)\} = \frac{s}{s^2-a^2};\; s > |a|$

$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s};\; s > 0$

$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathscr{L}\{\delta(t)\} = 1$

Inverse Laplace transforms of some basic functions

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$

$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t)$

$\mathscr{L}^{-1}\left\{e^{-st_0}\right\} = \delta(t-t_0)$

$\mathscr{L}^{-1}\{1\} = \delta(t)$

Properties of the Laplace and Inverse Laplace transform

Translation Theorem I:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a) = \mathscr{L}\{f(t)\}\big|_{s\to s-a}$$

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\left\{F(s)\big|_{s\to s-a}\right\} = e^{at}\mathscr{L}^{-1}\{F(s)\} = e^{at}f(t)$$

Translation Theorem II:

$$\mathscr{L}\{f(t-a)u_a(t)\} = e^{-as}F(s) = e^{-as}\mathscr{L}\{f(t)\}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t) = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}u_a(t)$$

Derivatives of Laplace Transforms:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathscr{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

Laplace Transform of Periodic Functions:
If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Laplace Transforms of Derivatives:

$$\begin{aligned}
\mathscr{L}\{y'\} &= sY(s) - y(0)\\
\mathscr{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0)\\
&\;\;\vdots\\
\mathscr{L}\{y^{(n)}(t)\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)
\end{aligned}$$

Convolution Theorem:

$$\mathscr{L}\{f(t)\star g(t)\} = F(s)G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)\star g(t)$$

$$(f\star g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau$$

Dirac Delta Function:

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$(f\star\delta)(t) = f(t)$$