

Homogeneous Linear Systems w/ Constant Coefficients (I)
- Real & Non-repeated Eigenvalues -

$$\textcircled{1} \begin{cases} \frac{dx}{dt} = -4x + y + z \\ \frac{dy}{dt} = x + 5y - z \\ \frac{dz}{dt} = y - 3z \end{cases}$$

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix}$$

$$= -(4+\lambda) \begin{vmatrix} 5-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= -(4+\lambda)(\lambda^2 - 2\lambda - 14) + 4 + \lambda = -\lambda^3 - 2\lambda^2 + 23\lambda + 60$$

Char. Eqn.: $\lambda^3 + 2\lambda^2 - 23\lambda - 60 = 0$

$$\lambda^3 + 3\lambda^2 - \lambda^2 - 3\lambda - 20\lambda - 60 = 0$$

$$(\lambda + 3)(\lambda^2 - \lambda - 20) = 0 \Rightarrow (\lambda + 3)(\lambda - 5)(\lambda + 4) = 0 \Rightarrow \text{eigenvalues: } -3, 5, -4$$

$\lambda = -3$:

$$[A + 3I | \vec{0}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{1/9 R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 = v_3 \\ v_2 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 5$:

$$[A - 5I | \vec{0}] = \left[\begin{array}{ccc|c} -9 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1/9 & -8/9 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/9 & -1/9 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 = v_3 \\ v_2 = 8v_3$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix}$$

$$\lambda = -4:$$

$$[A+4I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$v_1 = 10\sqrt{3}$$
$$v_2 = -\sqrt{3}$$

$$\vec{v}_3 = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$$

=> General Solution :

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} e^{-4t}$$

$$\textcircled{2} \quad \vec{x}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \vec{x}$$

$$\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{vmatrix} = -(10-\lambda)(12+\lambda) + 40$$
$$= \lambda^2 + 2\lambda - 80$$
$$= (\lambda-8)(\lambda+10) \Rightarrow \lambda = 8; -10$$

$$\lambda = 8:$$

$$[A-8I|\vec{0}] = \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array} \right] \xrightarrow{1/4 R_2} \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 2 & -5 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$v_1 = \frac{5}{2} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\lambda = -10:$$

$$[A+10I|\vec{0}] = \left[\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array} \right] \xrightarrow{\substack{1/2 R_2 \\ 1/5 R_1}} \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$v_1 = \frac{1}{4} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

=> General Solution :

$$\vec{x} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t}$$

$$\textcircled{3} \quad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

$$\lambda = 5: \quad [A - 5I | \vec{0}] = \begin{bmatrix} -4 & 2 & | & 0 \\ 4 & -2 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow v_1 = \frac{1}{2}v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda \in \{5, -1\}$$

$$\lambda = -1: \quad [A + I | \vec{0}] = \begin{bmatrix} 2 & 2 & | & 0 \\ 4 & 4 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{x} = c_1 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\textcircled{4} \quad \begin{cases} x'(t) = -4x + 2y \\ y'(t) = -\frac{5}{2}x + 2y \end{cases} \quad A = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 + 5 = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$$

$$\lambda = -3: \quad \begin{bmatrix} -1 & 2 & | & 0 \\ -5/2 & 5 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda \in \{-3, 1\}$$

$$\lambda = 1: \quad \begin{bmatrix} -5 & 2 & | & 0 \\ -5/2 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2/5 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow v_1 = +\frac{2}{5}v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ +5 \end{bmatrix}$$

$$\vec{x} = c_1 e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ +5 \end{bmatrix}$$

$$\textcircled{5} \begin{cases} x' = x + y - z \\ y' = 2y \\ z' = y - z \end{cases} \quad A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(\lambda-2)(\lambda+1) \Rightarrow \text{eigenvalues: } \lambda \in \{1, -1, 2\}$$

$$\lambda = 1:$$

$$[A - I | \vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right] \quad \begin{cases} v_2 = v_3 \\ v_2 = 0 \\ v_2 = 2v_3 \end{cases} \Rightarrow v_2 = v_3 = 0, v_1 \in \mathbb{R} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2:$$

$$[A - 2I | \vec{0}] = \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} v_1 &= 2v_3 \\ v_2 &= 3v_3 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\lambda = -1:$$

$$[A + I | \vec{0}] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} v_1 &= \frac{1}{2}v_3 \\ v_2 &= 0 \end{aligned}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

General Solution:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{-t}$$

$$\textcircled{6} \quad \vec{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = -(1+\lambda) \left(-(1+\lambda)(2-\lambda) - 3 \right) + (1+\lambda)$$

$$= (1+\lambda) (2+\lambda-\lambda^2+4) = (1+\lambda) (-\lambda^2+\lambda+6) = -(1+\lambda)(\lambda-3)(\lambda+2)$$

$$\Rightarrow \text{eigenvalues: } \lambda = -1; \lambda = 3; \lambda = -2$$

$$\lambda = -1: \quad [A + I | \vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right] \quad \begin{array}{l} v_2 = 0 \\ v_1 = -v_3 \end{array} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = 3: \quad [A - 3I | \vec{0}] = \left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/4 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/4 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & 1 & -4/3 & 0 \end{array} \right]$$

$$\begin{array}{l} v_1 = 1/4 v_2 \\ v_3 = 3/4 v_2 \end{array} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\lambda = -2: \quad [A + 2I | \vec{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} v_1 = -v_2 \\ v_3 = -3v_2 \end{array} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

General solution:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t}$$