HW15 Sol.

Linear Systems of First-Order with Constant Coefficients
II: Distinct Eigenvalues (with Complex Eigenvalues)

(1) $\vec{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29$$

$\Delta = 100 - 116 = -16$

$\lambda = \dfrac{10 \pm 4i}{2}$

$\boxed{\lambda = 5 \pm 2i}$

$\lambda = 5+2i$

$$[A-(5+2i)I \mid \vec{0}] = \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] \Rightarrow v_2 = (1-2i)v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{x} = c_1 e^{5t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin(2t)\right) + c_2 e^{5t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\cos(2t)\right)}$$

$$= c_1 e^{5t}\begin{bmatrix} \cos(2t) \\ \cos(2t)+2\sin(2t) \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} \sin(2t) \\ \sin(2t)-2\cos(2t) \end{bmatrix}.$$

(2) $\vec{x}' = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 + 5 = \lambda^2 - 8\lambda + 17$$

$\Delta = 64 - 68 = -4 \qquad \lambda = \dfrac{8 \pm 2i}{2}$

$\lambda = 4 \pm i$

$$\lambda = 4+i: \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right] \Rightarrow v_2 = (2-i)v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$\boxed{\vec{x} = c_1 e^{4t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) + c_2 e^{4t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t\right)}$$

$$= c_1 e^{4t}\begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$

(3) $\vec{x}' = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 + 2 = \lambda^2 - 8\lambda + 17 \qquad \lambda = 4 \pm i$$

$$\lambda = 4+i : \quad \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right] \Rightarrow v_2 = (-1+i)v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$$

$$\vec{x} = c_1 e^{4t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) + c_2 e^{4t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right)$$

$$= \boxed{c_1 e^{4t} \begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}}$$

(4) $\vec{x}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 - 16 + 25 = \lambda^2 + 9 \qquad \lambda = \pm 3i$$

$$\lambda = 3i : \quad \left[\begin{array}{cc|c} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \end{array}\right] \Rightarrow v_2 = \frac{4-3i}{5} v_1 \qquad \vec{v}_1 = \begin{bmatrix} 5 \\ 4-3i \end{bmatrix}$$

$$\vec{x} = c_1 e^0 \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} \cos(3t) - \begin{bmatrix} 0 \\ -3 \end{bmatrix} \sin(3t) \right) + c_2 e^0 \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} \sin(3t) + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cos(3t) \right)$$

$$= \boxed{c_1 \begin{bmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{bmatrix}}$$

5 $\begin{cases} x'(t) = z \\ y'(t) = -z \\ z'(t) = y \end{cases}$ $\qquad \vec{x}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2+1) \Rightarrow \lambda \in \{0, \pm i\}$$

$\lambda = 0$: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} v_3 = 0 \\ v_2 = 0 \end{array} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = i$: $\left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow[iR_2]{iR_1} \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{array}{l} v_1 = -i v_3 \\ v_2 = i v_3 \end{array} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ i \end{pmatrix}$

$$\vec{x} = c_1 e^0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^0 \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t \right) + c_3 e^0 \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t \right)$$

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix}}$$

6 $\vec{x}' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \vec{x}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((\lambda-1)^2-1\right)+2(1-\lambda)$$

$$= (1-\lambda)\left(\lambda^2-2\lambda+2\right)$$

$\lambda \in \{1, 1\pm i\}$

$\Delta = 4-8=-4 \Rightarrow \frac{2\pm 2i}{2}$

$\lambda = 1$:

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = 0$
$v_2 = 2v_3$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\lambda = 1+i$:

$$\left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right] \xrightarrow[-R_2,\ -R_3]{iR_1} \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 1 & i & 0 & 0 \\ 1 & 0 & i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & +2i & -2i & 0 \\ 0 & i & -i & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -i & 2i & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = -iv_3$
$v_2 = v_3$

$\vec{v}_2 = \begin{bmatrix} 1 \\ i \\ i \end{bmatrix}$

$$\vec{x} = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin t \right) + c_3 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos t \right)$$

$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -\sin t \\ -\sin t \end{bmatrix} + c_3 e^t \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}}$$

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$$\vec{x}' = \begin{pmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{pmatrix} \vec{x}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 5 & 1 \\ -5 & -6-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ -5 & -6-\lambda \end{vmatrix} = (2-\lambda)(\underbrace{\lambda^2+4\lambda+13}_{})$$

$$\Delta = 16-52 = -36$$

$$\lambda \in \left\{2, \frac{-4 \pm 6i}{2}\right\} \qquad \boxed{\lambda \in \{2, -2 \pm 3i\}}$$

$\lambda = 2$:

$$[A-2I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ -5 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 8/5 & -4/5 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -28/25 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v_1 = \frac{28}{25} v_3 \; ; \; v_2 = -\frac{1}{5} v_3 \qquad \boxed{\vec{v}_1 = \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix}}$$

$\lambda = -2-3i$

$$[A+(2+3i)I|\vec{0}] = \left[\begin{array}{ccc|c} 4+3i & 5 & 1 & 0 \\ -5 & -4+3i & 4 & 0 \\ 0 & 0 & 4+3i & 0 \end{array}\right] \begin{array}{l} \Rightarrow v_1 = -\dfrac{5}{4+3i} v_2 \\ \qquad \Uparrow \\ \Rightarrow v_3 = 0 \end{array}$$

$$\vec{v}_2 = \begin{bmatrix} 5 \\ -4-3i \\ 0 \end{bmatrix}$$

$$\boxed{\vec{x} = c_1 \begin{bmatrix} 28 \\ -5 \\ 25 \end{bmatrix} e^{2t} + c_2 e^{-2t} \begin{bmatrix} 5\cos(3t) \\ -4\cos(3t) + 3\sin(3t) \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 5\sin(3t) \\ -4\sin(3t) - 3\cos(3t) \\ 0 \end{bmatrix}}$$

Repeated Eigenvalues

8 $\vec{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2)+1 = \lambda^2-6\lambda+9 = (\lambda-3)^2$$

$\lambda = 3$ $\rightarrow$ algebraic multiplicity 2

$$[A-3I \mid \vec{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow v_1 = v_2 \quad \boxed{\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \Rightarrow \boxed{\vec{x}_1 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\hookrightarrow$ geometric multiplicity 1

Second Solution?

$$(A-3I)\vec{w} = \vec{v} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow w_1 - w_2 = 1 \Rightarrow w_1 = w_2 + 1$$

$$\vec{w} = \begin{pmatrix} w_2+1 \\ w_2 \end{pmatrix} \quad \boxed{\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\boxed{\vec{x}_2 = t e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\Rightarrow \boxed{\vec{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left[t e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]}$$

or

$$\Rightarrow \boxed{\vec{x} = e^{3t}\begin{pmatrix} c_1 + c_2 t + c_2 \\ c_1 + c_2 t \end{pmatrix}}$$

9 $\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 21 + 4 = (\lambda-5)^2$$

$$[A - 5I \mid \vec{0}] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] \qquad 2v_1 + v_2 = 0 \qquad v_2 = -2v_1 \qquad \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boxed{\vec{x}_1 = e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

Second Solution:

$$(A - 5I)\vec{w} = \vec{v} \qquad \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad 2w_1 + w_2 = 1$$

Take $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}_2 = e^{5t} \cdot t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$\boxed{\vec{x} = c_1 e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left[te^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]} \text{ or } \boxed{\vec{x} = e^{5t}\begin{pmatrix} c_1 + tc_2 \\ -2c_1 - 2tc_2 + c_2 \end{pmatrix}}$$

10 $\vec{x}' = \begin{pmatrix} -1 & 3/2 \\ -1/6 & -2 \end{pmatrix} \vec{x}$

$$\begin{vmatrix} -1-\lambda & 3/2 \\ -1/6 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 1 + \frac{1}{4} = \lambda^2 + 3\lambda + \frac{9}{4} = (\lambda + 3/2)^2 \qquad \lambda = -3/2$$

$$[A + 3/2\, I \mid \vec{0}] = \left[\begin{array}{cc|c} 1/2 & 3/2 & 0 \\ -1/6 & -1/2 & 0 \end{array}\right] \qquad v_1 + 3v_2 = 0 \Rightarrow \boxed{\vec{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}} \Rightarrow \vec{x}_1 = e^{-3/2 t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$(A + 3/2\, I)\vec{w} = \vec{v}$$

$$\begin{pmatrix} 1/2 & 3/2 \\ -1/6 & -1/2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow \frac{1}{2}w_1 + \frac{3}{2}w_2 = -3$$

$(w_2 = 2)$: $\boxed{\vec{w} = \begin{pmatrix} -12 \\ 2 \end{pmatrix}}$ or $(w_2 = 0)$: $\boxed{\vec{w} = \begin{pmatrix} -6 \\ 0 \end{pmatrix}}$ etc.

$$\Rightarrow \boxed{\vec{x}_2 = e^{-3/2 t} \cdot t\begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-3/2 t}\begin{pmatrix} -12 \\ 2 \end{pmatrix}}$$

$$\Rightarrow \boxed{\vec{x} = e^{-3/2 t}\begin{pmatrix} -3c_1 - 3c_2 t - 12c_2 \\ c_1 + c_2 t + 2c_2 \end{pmatrix}}$$

11 $\vec{x}' = \begin{pmatrix} 2 & 1/2 \\ -1/2 & 1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 1/2 \\ -1/2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 + \frac{1}{4} = (\lambda - 3/2)^2$$

$$\left[A - 3/2 I \mid \vec{0}\right] = \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 0 \end{array}\right] \qquad v_1 = -v_2 \qquad \boxed{\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$(A - \lambda I)\vec{w} = \vec{v} \qquad \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \frac{1}{2}(w_1 + w_2) = -1$$
$$w_1 + w_2 = -2 \qquad \boxed{\vec{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

$$\boxed{\begin{aligned} \vec{x} &= c_1 e^{3/2 t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left(t e^{3/2 t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{3/2 t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \\ &= e^{3/2 t}\begin{pmatrix} -c_1 - c_2 - c_2 t \\ c_1 - c_2 + c_2 t \end{pmatrix} \end{aligned}}$$

12 $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} \overset{R_1+R_2}{=} \begin{vmatrix} -1-\lambda & -1-\lambda & 0 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} \lambda+1 & \lambda+1 & 0 \\ 2 & \lambda-1 & 2 \\ 2 & -2 & 1-\lambda \end{vmatrix} \underset{C_2 - C_1}{=} \begin{vmatrix} \lambda+1 & 0 & 0 \\ 2 & \lambda-3 & 2 \\ 2 & -4 & 1-\lambda \end{vmatrix}$$

$$= (\lambda+1)(-\lambda^2 + 4\lambda - 3 + 8) = (\lambda+1)(-\lambda^2 + 4\lambda + 5)$$
$$= -(\lambda+1)(\lambda-5)(\lambda+1)$$
$$= -(\lambda+1)^2(\lambda-5)$$

Eigenvalues: $\lambda_1 = -1$; $\lambda_2 = -1$; $\lambda_3 = 5$.

$\Rightarrow \boxed{alg(\lambda=5) = geom(\lambda=5) = 1}$

$\Rightarrow \boxed{alg(\lambda=-1) = 2}$

$$[A+I|\vec{0}] = \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_1+R_2} \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 - v_2 + v_3 = 0$

(3 unknowns, 1 restriction) => we can find $\underline{2}$ linearly indp. eigenvectors

For example:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \ \& \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \boxed{geom(\lambda=-1) = 2}$$

=> Matrix is non-defective

Eigenvector for $\lambda = 5$:

$$[A-5I|\vec{0}] = \left[\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ +2 & -2 & -4 & 0 \end{array}\right] \xrightarrow[\substack{-\frac{1}{2}R_2 \\ \frac{1}{2}R_3}]{-\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 2 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow[\frac{2}{3}R_2]{R_3+R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_1 = v_3 \\ v_2 = -v_3 \end{array}$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Solution to $\vec{x}' = A\vec{x}$:

$$\boxed{\vec{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

(13) $A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ Eigenvalues: $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 2}$ $\Rightarrow \boxed{\text{alg}(\lambda=2) = 3}$

$$[A-2I|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow v_2 + 6v_3 = 0 \Rightarrow v_2 = 0 \\ \Rightarrow v_3 = 0 \end{array} \Rightarrow \boxed{\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$\Rightarrow \boxed{\text{geom}(\lambda=2) = 1} \Rightarrow$ defective matrix

(14) $A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \underset{R_1-R_2}{\overset{R_3-R_2}{=}} \begin{vmatrix} 2-\lambda & \lambda-2 & 0 \\ 1 & 1-\lambda & -1 \\ 0 & \lambda-2 & \lambda-2 \end{vmatrix} \overset{C_2+C_1}{=} \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & \lambda-2 & \lambda-2 \end{vmatrix}$$

$$= (2-\lambda)\left(-(\lambda-2)^2 + (\lambda-2)\right) = (\lambda-2)(\lambda-2)\left(\lambda-2+1\right) = (\lambda-2)^2(\lambda-1)$$

$\boxed{\lambda_1 = \lambda_2 = 2}$; $\boxed{\lambda_3 = 1}$ $\Rightarrow \boxed{\text{alg}(\lambda=1) = \text{geom}(\lambda=1) = 1}$; $\boxed{\text{alg}(\lambda=2) = 2}$

$$[A-2I|\vec{0}] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 - v_2 - v_3 = 0$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\boxed{\text{geom}(\lambda=2) = 2} \Rightarrow$ non-defective

$$[A-I|\vec{0}] = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} v_1 = v_3 \\ v_2 = v_3 \end{array} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution to $\vec{x}' = A\vec{x}$: $\boxed{\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$