Separable Equations

1) $\frac{dy}{dx} = \sin(5x)$

$$dy = \sin(5x)\,dx$$

Integrate both sides : $\int dy = \int \sin(5x)\,dx$

$$y = -\frac{1}{5}\cos(5x) + C$$

$$\boxed{y = -\frac{1}{5}\cos(5x) + C}$$

2) $xy' = 4y$

$$x\frac{dy}{dx} = 4y \Rightarrow \frac{1}{4y}dy = \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{4}\ln|y| = \ln|x| + C$$

$$\Rightarrow e^{\frac{1}{4}\ln|y|} = e^{\ln|x|+C}$$

$$\Rightarrow |y|^{1/4} = |x|e^{C} \Rightarrow |y| = C|x|^4$$

$$\Rightarrow y = \pm C|x|^4$$

$$\boxed{y = Cx^4}$$

Remark : we divided by y
Is $y=0$ a valid solution? Yes
Did we lose a solution? No
because $y=0$ is represented in $y = Cx^4$

3) $(4y + yx^2)\,dy - (2x + xy^2)\,dx = 0$

$$y(4+x^2)\,dy = x(2+y^2)\,dx$$

$$\frac{y}{2+y^2}dy = \frac{x}{4+x^2}dx$$

$$\frac{1}{2}\ln|2+y^2| = \frac{1}{2}\ln|4+x^2| + C$$

$$\ln(2+y^2) = \ln(4+x^2) + C$$

$$e^{\ln(2+y^2)} = e^{\ln(4+x^2)+C}$$

$$2+y^2 = (4+x^2)e^{C}$$

$$\boxed{2+y^2 = C(4+x^2)}$$

Any issues when dividing?
No, because both $(2+y^2), (4+x^2) > 0$

4. $$\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{(y+3)-5}{y+3}\,dy = \int\left(1-\frac{5}{y+3}\right)dy = y-5\ln|y+3|+C$$

$$\int \frac{x-1}{x+4}\,dx = \int \frac{(x+4)-5}{x+4}\,dx = \int\left(1-\frac{5}{x+4}\right)dx = x-5\ln|x+4|+C$$

$$\boxed{y-5\ln|y+3| = x-5\ln|x+4|+C}$$ ← perfectly good answer

or you can simplify further:

$$(y-x)+C = 5\ln|y+3|-5\ln|x+4|$$
$$= 5\ln\left|\frac{y+3}{x+4}\right|$$

$$\Rightarrow e^{(y-x)+C} = \left|\frac{y+3}{x+4}\right|^5$$

$$\Rightarrow C e^{(y-x)} = \pm\left(\frac{y+3}{x+4}\right)^5 \quad \Rightarrow \boxed{e^{y-x} = C\left(\frac{y+3}{x+4}\right)^5}$$

5. $y\,dy = 4x\sqrt{y^2+1}\,dx$; $y(0)=1$

$$\int \frac{y}{\sqrt{y^2+1}}\,dy = \int 4x\,dx$$

$$\sqrt{y^2+1} = 2x^2+C$$

$x=0, y=1$: $\sqrt{2} = C \Rightarrow \boxed{\sqrt{y^2+1} = 2x^2+\sqrt{2}}$

(6) $\frac{dx}{dy} = 4(x^2+1)$; $x(\pi/4) = 1.$

$\int \frac{1}{x^2+1}\,dx = \int 4\,dy \Rightarrow \arctan(x) = 4y + C$

$x = 1 ;\ y = \pi/4 \Rightarrow \arctan(1) = \pi + C$

$\pi/4 = \pi + C \Rightarrow C = -3\pi/4$

$\Rightarrow \arctan(x) = 4y - 3\pi/4$

or $\boxed{x = \tan(4y - 3\pi/4)}$

(7) $x^2 y' = y - xy$; $y(-1) = -1.$

$x^2 \frac{dy}{dx} = y(1-x)$

$\int \frac{1}{y}\,dy = \int \frac{1-x}{x^2}\,dx$

$\ln|y| = -\frac{1}{x} - \ln|x| + C$

$\ln|xy| = -\frac{1}{x} + C$

$|xy| = e^{-\frac{1}{x}+C} \Rightarrow \boxed{xy = ce^{-\frac{1}{x}}}$

$x = y = -1 \Rightarrow 1 = ce \Rightarrow c = e^{-1} \Rightarrow \boxed{xy = e^{-\frac{1}{x}-1}}$

(8) $e^y \sin(2x)\,dx + \cos x\,(e^{2y} - y)\,dy = 0$

$e^y \sin(2x)\,dx = \cos x\,(y - e^{2y})\,dy$

$\int \frac{\sin(2x)}{\cos x}\,dx = \int \frac{y - e^{2y}}{e^y}\,dy$

$\int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\,dx$

$= -2\cos x + C$

$\int ye^{-y} - e^y\,dy = -\int y(e^{-y})'\,dy - e^y + C$

$= -ye^{-y} + \int e^{-y}\,dy - e^{+y} + C$

$= -ye^{-y} - e^{-y} - e^y + C$

$\Rightarrow \boxed{ye^{-y} + e^{-y} + e^y = 2\cos x + C}$

9) $\frac{dy}{dx} = (x+y+1)^2$

$u = x+y+1 \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{du}{dx} - 1 = u^2 \Rightarrow \int \frac{1}{1+u^2}\, du = \int dx$

$\Rightarrow \arctan(u) = x + c$

$\Rightarrow u = \tan(x+c)$

$\Rightarrow x+y+1 = \tan(x+c) \Rightarrow \boxed{y = \tan(x+c) - x - 1}$

10) $\frac{dy}{dx} = 1 + e^{y-x+5}$

$u = y-x+5 \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 1$

$\frac{du}{dx} + \cancel{1} = \cancel{1} + e^u \Rightarrow \int e^{-u}\, du = \int dx \Rightarrow -e^{-u} = x + c$

$\Rightarrow e^{-u} = c - x \Rightarrow -u = \ln(c-x)$

$\Rightarrow x - y - 5 = \ln(c-x)$

$\Rightarrow \boxed{y = x - 5 - \ln(c-x)}$

11) $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

$u = y-2x+3 \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 2$

$\frac{du}{dx} + 2 = 2 + \sqrt{u} \Rightarrow \int \frac{1}{\sqrt{u}}\, du = \int dx \Rightarrow 2\sqrt{u} = x + C$

$\Rightarrow \sqrt{u} = \frac{1}{2}x + C \Rightarrow u = \left(\frac{1}{2}x + c\right)^2$

$\Rightarrow y - 2x + 3 = \left(\frac{1}{2}x + c\right)^2$

or: $u = \left(\frac{1}{2}x + c\right)^2 = \left(\frac{x+2c}{2}\right)^2$ $\Rightarrow \boxed{y = 2x - 3 + \left(\frac{1}{2}x + c\right)^2}$

relabel: $u = \frac{(x+c)^2}{4} \Rightarrow 4u = (x+c)^2$

$\Rightarrow \boxed{4(y - 2x + 3) = (x+c)^2}$