

Linear First Order ODEs - Integrating Factors

① $x \frac{dy}{dx} - 4y = x^6 e^x$

Standard Form: $\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$

$$p(x) = -\frac{4}{x}$$

$$\int p(x) dx = -4 \ln|x| + c \Rightarrow \mu(x) = e^{-4 \ln|x|} = \frac{1}{x^4}$$

Multiply Equation by $\frac{1}{x^4}$:

$$\frac{1}{x^4} \frac{dy}{dx} - \frac{4}{x^5} y = x e^x$$

$$\frac{d}{dx} \left[\frac{1}{x^4} y \right] = x e^x \Rightarrow \frac{1}{x^4} y = \int x e^x dx = x e^x - e^x + c$$

$$\Rightarrow \boxed{y = x^5 e^x - x^4 e^x + c x^4}$$
 Interval of validity:
 $(-\infty, \infty)$

② $\frac{dy}{dx} + 2xy = x$; $y(0) = -3$

$$p(x) = 2x; \int 2x dx = x^2 + c; \mu(x) = e^{x^2}$$

$$e^{x^2} \frac{dy}{dx} + 2x e^{x^2} y = x e^{x^2} \Rightarrow \frac{d}{dx} (e^{x^2} y) = x e^{x^2}$$

$$\Rightarrow e^{x^2} y = \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$$

$$\Rightarrow \boxed{y = \frac{1}{2} + c e^{-x^2}}$$

$$x=0, y=-3 \Rightarrow -3 = \frac{1}{2} + c \Rightarrow c = -\frac{7}{2}$$

$$\boxed{y = \frac{1}{2} - \frac{7}{2} e^{-x^2}}$$
 Interval of validity:
 $(-\infty, \infty)$

$$\textcircled{3} \quad y' + 3x^2y = x^2$$

$$p(x) = 3x^2 \Rightarrow \mu(x) = e^{x^3}$$

$$\frac{d}{dx}(e^{x^3}y) = x^2e^{x^3} \Rightarrow e^{x^3}y = \frac{1}{3}e^{x^3} + c \Rightarrow \boxed{y = \frac{1}{3} + ce^{-x^3}}$$

Interval of validity: $(-\infty, \infty)$.

$$\textcircled{4} \quad \frac{dy}{dx} + 5y = 20; \quad y(0) = 2.$$

$$p(x) = 5 \Rightarrow \mu(x) = e^{5x}$$

$$\frac{d}{dx}(e^{5x}y) = 20e^{5x} \Rightarrow e^{5x}y = 4e^{5x} + c \Rightarrow y = 4 + ce^{-5x}$$

$$x=0, y=2 \Rightarrow 2 = 4 + c \Rightarrow \boxed{y = 4 - 2e^{-5x}} \quad x \in (-\infty, \infty)$$

$$\textcircled{5} \quad xdy = (x \sin x - y) dx$$

$$\frac{dy}{dx} = \sin x - \frac{1}{x}y$$

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x; \quad p(x) = \frac{1}{x}; \quad \int p(x) dx = \ln|x| + c$$

Take $x \in (0, \infty)$. Then $\mu(x) = x$, so

$$\frac{d}{dx}(xy) = x \sin x$$

$$\Rightarrow xy = \int x \sin x dx$$

$$= -x \cos x + \sin x + c$$

$$\Rightarrow \boxed{y = -\cos x + \frac{1}{x} \sin x + \frac{c}{x}} \quad x \in (0, \infty)$$

Remark:

The expression of the solution is the same on $(-\infty, 0)$.

$$\textcircled{6} (1+e^x) \frac{dy}{dx} + e^x y = 0$$

$$\frac{dy}{dx} + \frac{e^x}{1+e^x} y = 0$$

$$p(x) = \frac{e^x}{1+e^x} \Rightarrow \int p(x) dx = \ln(1+e^x) + C$$

$$\Rightarrow \mu(x) = 1+e^x$$

$$\Rightarrow \frac{d}{dx} [(1+e^x)y] = 0 \Rightarrow (1+e^x)y = C$$

$$\Rightarrow \boxed{y = \frac{C}{1+e^x}} \quad x \in (-\infty, \infty)$$

$$\textcircled{7} x \frac{dy}{dx} + 4y = x^3 - x$$

$$\frac{dy}{dx} + \frac{4}{x} y = x^2 - 1$$

$$p(x) = \frac{4}{x}; \mu(x) = x^4; \frac{d}{dx} (x^4 y) = x^4 (x^2 - 1)$$

$$x^4 y = \frac{1}{7} x^7 - \frac{1}{5} x^5 + C$$

$$\boxed{y = \frac{1}{7} x^3 - \frac{1}{5} x + Cx^{-4}} \quad x \in (0, \infty)$$

$$\textcircled{8} (x+1) \frac{dy}{dx} + y = \ln x; y(1) = 10$$

$$\frac{d}{dx} [(x+1)y] = \ln x$$

$$(x+1)y = \int \ln x dx \\ = x \ln x - x + C$$

$$x=1; y=10$$

$$\Rightarrow 20 = -1 + C \Rightarrow C = 21$$

$$\boxed{(x+1)y = x \ln x - x + 21}$$

$$x \in (0, \infty)$$

$$\textcircled{9} (x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy$$

$$(x+2)^2 \frac{dy}{dx} + 4y(x+2) = 5$$

$$\frac{dy}{dx} + \frac{4}{x+2} y = \frac{5}{(x+2)^2}$$

$$p(x) = \frac{4}{x+2}; \int p(x) dx = 4 \ln|x+2| + C$$

$$\text{Take } x \in (-2, \infty): \mu(x) = e^{4 \ln(x+2)} = (x+2)^4$$

$$\Rightarrow \frac{d}{dx} [(x+2)^4 y] = 5(x+2)^2$$

$$\Rightarrow (x+2)^4 y = \frac{5}{3} (x+2)^3 + C$$

$$\Rightarrow \boxed{y = \frac{5}{3} (x+2)^{-1} + C(x+2)^{-4}} \quad x \in (-2, \infty)$$

$$(10) \quad \frac{dy}{dx} + y = \frac{1 - e^{-2x}}{e^x + e^{-x}}$$

$$p(x) = 1; \quad \mu(x) = e^x$$

$$\frac{d}{dx}(e^x y) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$e^x y = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$= \ln(e^x + e^{-x}) + C$$

$$y = e^{-x} \ln(e^x + e^{-x}) + C e^{-x}$$

$$x \in (-\infty, \infty)$$

$$(11) \quad x^2 y' + x(x+2)y = e^x$$

$$y' + \frac{x+2}{x} y = \frac{1}{x^2} e^x$$

$$p(x) = \frac{x+2}{x}; \quad \int p(x) dx = \int \left(1 + \frac{2}{x}\right) dx$$

$$= x + 2 \ln(x) \quad \text{on } (0, \infty)$$

$$\Rightarrow \mu(x) = e^{x+2 \ln(x)} = x^2 e^x$$

$$\frac{d}{dx}(x^2 e^x y) = e^{2x} \Rightarrow x^2 e^x y = \frac{1}{2} e^{2x} + C$$

$$y = \frac{1}{2x^2} e^x + \frac{C}{x^2} e^{-x}$$

$$x \in (0, \infty)$$

$$(12) \quad x \frac{dy}{dx} + (3x+1)y = e^{-3x}$$

$$\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)y = \frac{1}{x} e^{-3x}$$

$$p(x) = 3 + \frac{1}{x}; \quad \int p(x) dx = 3x + \ln(x)$$

$$\text{on } (0, \infty)$$

$$\mu(x) = x e^{3x}$$

$$\frac{d}{dx}(x e^{3x} y) = 1 \Rightarrow x e^{3x} y = x + C$$

$$y = e^{-3x} + \frac{C}{x} e^{-3x}$$

$$x \in (0, \infty)$$

$$(13) \quad x(x-2)y' + 2y = 0; \quad y(3) = 6$$

$$y' + \frac{2}{x(x-2)} y = 0$$

$$p(x) = \frac{2}{x(x-2)} = \frac{x - (x-2)}{x(x-2)} = \frac{1}{x-2} - \frac{1}{x}$$

Take $x \in (2, \infty) \rightarrow$ look at initial condition
($x=3, y=6$)

$$\int p(x) dx = \ln|x-2| - \ln|x|$$

$$= \ln(x-2) - \ln x = \ln\left(\frac{x-2}{x}\right)$$

$$\Rightarrow \mu(x) = \frac{x-2}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x-2}{x} y\right) = 0$$

$$\Rightarrow \frac{x-2}{x} y = C \Rightarrow y = \frac{Cx}{x-2}$$

$$x=3, y=6 \Rightarrow 6 = 3C \Rightarrow C=2$$

$$y = \frac{2x}{x-2}; \quad x \in (2, \infty)$$

$$(14) \quad \frac{dy}{dx} = \frac{y}{y-x}; \quad y(5)=2$$

Remark: This is not separable, and not linear in y

But look at the reciprocal:

$$\frac{dx}{dy} = \frac{y-x}{y} \Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{y}x = 1} \quad (*)$$

↳ This is linear in x

So apply integrating factors:

$$p(y) = \frac{1}{y} \Rightarrow \int p(y) dy = \ln(y)$$

↳ Take $y \in (0, \infty)$

$$\Rightarrow \mu(y) = y$$

$$\Rightarrow (*) \text{ becomes: } y \frac{dx}{dy} + x = y$$

$$\frac{d}{dy}(yx) = y$$

$$yx = \int y dy = \frac{1}{2}y^2 + C$$

$$\boxed{x = \frac{1}{2}y + \frac{C}{y}}$$

$$\text{I.C.: } x=5, y=2 : 5 = 1 + \frac{C}{2} \Rightarrow C=8$$

$$\boxed{x = \frac{1}{2}y + \frac{8}{y}} \quad y \in (0, \infty).$$

$$(15) (x+4y^2)dy + 2ydx = 0$$

Bring to standard form: $(x+4y^2) \frac{dy}{dx} + 2y = 0$

$$\frac{dy}{dx} + \frac{2y}{x+4y^2} = 0$$

↳ not linear in y!

So try the same reciprocal trick:

$$\frac{dy}{dx} = -\frac{2y}{x+4y^2} \Rightarrow \frac{dx}{dy} = -\frac{x+4y^2}{2y} = -\frac{1}{2y}x - 2y$$

$$\Rightarrow \boxed{\frac{dx}{dy} + \frac{1}{2y}x = -2y} \quad \text{linear in } x \quad \checkmark$$

$$p(y) = \frac{1}{2y}; \int p(y)dy = \frac{1}{2} \ln y \Rightarrow \mu(y) = \sqrt{y}$$

$y \in (0, \infty)$

$$\Rightarrow \frac{d}{dy}(\sqrt{y} \cdot x) = -2y\sqrt{y} \Rightarrow \sqrt{y} \cdot x = -2 \int y^{3/2} dy$$
$$= -2 \cdot \frac{2}{5} y^{5/2} + c$$

$$\Rightarrow \boxed{x = -\frac{4}{5}y^2 + \frac{c}{\sqrt{y}}} \quad y \in (0, \infty)$$

$$(16) \quad y dx + (xy + 2x - ye^y) dy = 0$$

→ this guy should already tell you that this will not be linear in y .

$$y \frac{dx}{dy} + xy + 2x - ye^y = 0$$

$$\frac{dx}{dy} + \frac{y+2}{y} x = e^y$$

$$p(y) = \frac{y+2}{y} = 1 + \frac{2}{y} \Rightarrow \int p(y) dy = y + 2 \ln(y)$$

$y \in (0, \infty)$

$$\Rightarrow \mu(y) = y^2 e^y$$

$$\frac{d}{dy} (y^2 e^y x) = y^2 e^{2y}$$

$$y^2 e^y x = \int y^2 e^{2y} dy$$

$$= \frac{1}{2} \int y^2 (e^{2y})' dy = \frac{1}{2} y^2 e^{2y} - \int y e^{2y} dy$$

$$= \frac{1}{2} y^2 e^{2y} - \frac{1}{2} y e^{2y} + \frac{1}{4} e^{2y} + c$$

$$x = \frac{1}{2} e^y - \frac{1}{2y} e^y + \frac{1}{4y^2} e^y + \frac{c}{y^2} e^{-y} \quad y \in (0, \infty).$$