Homework 4 solutions - Exact Equations

1 $(5x+4y)dx + (4x-8y^3)dy = 0$

$M(x,y) = 5x+4y \Rightarrow \frac{\partial M}{\partial y} = 4$ ✓

$N(x,y) = 4x-8y^3 \Rightarrow \frac{\partial N}{\partial x} = 4$ ✓

Exact

Find $f(x,y)$ such that $\frac{\partial f}{\partial x} = 5x+4y$; $\frac{\partial f}{\partial y} = 4x-8y^3$

$\frac{\partial f}{\partial x} = 5x+4y \Rightarrow f(x,y) = 5\frac{x^2}{2} + 4xy + g(y)$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= 4x + g'(y)\\ &= 4x - 8y^3\end{aligned}\right\} \Rightarrow g'(y) = -8y^3 \Rightarrow g(y) = -2y^4$$

$\Rightarrow \boxed{f(x,y) = 5\frac{x^2}{2} + 4xy - 2y^4}$

$\Rightarrow$ Solutions to the ODE : $\boxed{\frac{5x^2}{2} + 4xy - 2y^4 = C}$

2 $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

$M(x,y) = \sin y - y\sin x \Rightarrow \frac{\partial M}{\partial y} = \cos y - \sin x$ ✓

$N(x,y) = \cos x + x\cos y - y \Rightarrow \frac{\partial N}{\partial x} = -\sin x + \cos y$ ✓

Exact

Find $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = \sin y - y\sin x$; $\frac{\partial f}{\partial y} = \cos x + x\cos y - y$

$\frac{\partial f}{\partial x} = \sin y - y\sin x \Rightarrow f(x,y) = x\sin y + y\cos x + g(y)$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= x\cos y + \cos x + g'(y)\\ &= x\cos y + \cos x - y\end{aligned}\right\} \Rightarrow g'(y) = -y \Rightarrow g(y) = -y^2/2$$

$\Rightarrow \boxed{f(x,y) = x\sin y + y\cos x - y^2/2}$

$\Rightarrow$ Solutions to the ODE : $\boxed{x\sin y + y\cos x - y^2/2 = C}$

(3) $\underbrace{[\cos(xy) - xy\sin(xy)]}_{M(x,y)}dx \underbrace{- x^2\sin(xy)}_{N(x,y)}\,dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin(xy) - x\sin(xy) - x^2 y\cos(xy)$$
$$= -2x\sin(xy) - x^2 y\cos(xy) \checkmark$$

<u>Exact</u>

$$\frac{\partial N}{\partial x} = -2x\sin(xy) - x^2 y\cos(xy) \checkmark$$

Find potential: $\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial x} = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial f}{\partial y} = -x^2\sin(xy)$$ ⇝ easier to integrate dy

$\hookrightarrow \Rightarrow f(x,y) = x\cos(xy) + g(x)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial x} &= \cos(xy) - xy\sin(xy) + g'(x) \\ &= \cos(xy) - xy\sin(xy) \end{aligned}\right\} \Rightarrow g'(x) = 0 \Rightarrow g(x) = C$$

$\Rightarrow \boxed{f(x,y) = x\cos(xy)}$

$\Rightarrow$ Solution to ODE: $\boxed{x\cos(xy) = C}$

(4) $ye^{xy}dx + (2y - xe^{xy})dy = 0$

$$M(x,y) = ye^{xy} \Rightarrow \frac{\partial M}{\partial y} = e^{xy} + xye^{xy}$$

$$N(x,y) = 2y - xe^{xy} \Rightarrow \frac{\partial N}{\partial x} = -e^{xy} - xye^{xy}$$

NOT Exact

(5) $(1 + \ln(xy))dx + xy^{-1}dy = 0$

$$M(x,y) = 1 + \ln(xy) \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{xy} \cdot x = \frac{1}{y} \checkmark$$

$$N(x,y) = \frac{x}{y} \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{y} \checkmark$$

Exact

Find potential $f(x,y)$:

$$\frac{\partial f}{\partial x} = 1 + \ln(xy)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} \rightsquigarrow \Rightarrow f(x,y) = x\ln y + g(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \ln y + g'(x) = 1 + \ln(xy)$$

$$\Rightarrow \ln y + g'(x) = 1 + \ln(xy)$$

$$\Rightarrow g'(x) = 1 + \ln(xy) - \ln y = 1 + \ln x$$

$$\Rightarrow g(x) = x + x\ln x - x = x\ln x$$

$$\Rightarrow f(x,y) = x\ln y + x\ln x = x\ln(xy)$$

$\Rightarrow$ Solutions to ODE: $\boxed{x\ln(xy) = C}$

⑥ $(2y^2x-3)dx+(2yx^2+4)dy=0$

$\frac{\partial M}{\partial y} = 4yx$ ✓

$\frac{\partial N}{\partial x} = 4yx$ ✓

<u>Exact</u>

<u>Potential</u>: $\frac{\partial f}{\partial x} = 2y^2x-3 \Rightarrow f(x,y) = x^2y^2-3x+g(y)$

$\frac{\partial f}{\partial y} = 2yx^2+4$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y} &= 2x^2y+g'(y)\\ &= 2x^2y+4\end{aligned}\right\} \Rightarrow g'(y)=4 \Rightarrow g(y)=4y$$

$\Rightarrow \boxed{f(x,y)=x^2y^2-3x+4y}$

$\Rightarrow$ Solution to ODE: $\boxed{x^2y^2-3x+4y=C}$

⑦ $\underbrace{\left(2y-\frac{1}{x}+\cos(3x)\right)}_{N(x,y)}\frac{dy}{dx}+\underbrace{\frac{y}{x^2}-4x^3+3y\sin(3x)}_{M(x,y)}=0$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x^2}+3\sin(3x)\\ \frac{\partial N}{\partial x} &= +\frac{1}{x^2}-3\sin(3x)\end{aligned}\right\} \Rightarrow \underline{\text{not}}\ \text{exact.}$$

⑧ $(x^3+y^3)dx+3xy^2dy=0$

$\frac{\partial M}{\partial y} = 3y^2$; $\frac{\partial N}{\partial x} = 3y^2 \Rightarrow$ <u>Exact</u>

$\frac{\partial f}{\partial x} = x^3+y^3 \Rightarrow f(x,y)=\frac{1}{4}x^4+y^3x+g(y)$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y} &= 3y^2x+g'(y)\\ \frac{\partial f}{\partial y} &= 3xy^2\end{aligned}\right\} \Rightarrow g'(y)=0 \Rightarrow g(y)=c$$

$\boxed{f(x,y)=\frac{1}{4}x^4+y^3x}$ $\Rightarrow$ Solution to ODE: $\boxed{\frac{1}{4}x^4+y^3x=C}$

9. $(y^3 - y^2\sin x - x)\,dx + (3xy^2 + 2y\cos x)\,dy = 0$

$$\frac{\partial M}{\partial y} = 3y^2 - 2y\sin x \quad ✓$$

$$\frac{\partial N}{\partial x} = 3y^2 - 2y\sin x \quad ✓$$

Exact

Potential: $\frac{\partial f}{\partial x} = y^3 - y^2\sin x - x$

$$\Rightarrow f(x,y) = y^3x + y^2\cos x - \frac{x^2}{2} + g(y)$$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y} &= 3y^2x + 2y\cos x + g'(y)\\ &= 3xy^2 + 2y\cos x\end{aligned}\right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$$

$$\boxed{f(x,y) = y^3x + y^2\cos x - \frac{x^2}{2}} \Rightarrow \text{Solution to ODE: } \boxed{y^3x + y^2\cos x - \frac{x^2}{2} = c}$$

10. $\left(y\ln y - e^{-xy}\right)dx + \left(\frac{1}{y} + x\ln y\right)dy = 0$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= \ln y + 1 + xe^{-xy}\\ \frac{\partial N}{\partial x} &= \ln y\end{aligned}\right\} \text{Not exact.}$$

11. $\frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy = 0$

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2}\ ; \quad \frac{\partial N}{\partial x} = -\frac{2x}{y^2} \quad \Rightarrow \text{Exact}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{1}{y}x^2 + g(y)$$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial y} &= -\frac{x^2}{y^2} + g'(y)\\ &= -\frac{x^2}{y^2}\end{aligned}\right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$$

$$\boxed{f(x,y) = \frac{1}{y}x^2} \Rightarrow \text{Solution to ODE: } \boxed{\frac{1}{y}x^2 = c} \text{ or } \boxed{y = cx^2}$$

Remark: #11 is also separable:

$$\frac{2x}{y}dx = \frac{x^2}{y^2}dy \Rightarrow \frac{2}{x}dx = \frac{1}{y}dy \Rightarrow 2\ln|x| + c = \ln|y|$$

$$\Rightarrow y = \pm e^c x^2 \Rightarrow \boxed{y = cx^2} \; (c \neq c$$

12 $\left(\frac{1}{x} - \frac{y}{x^2+y^2}\right)dx + \frac{x}{x^2+y^2}dy = 0$

$$\frac{\partial M}{\partial y} = -\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} \checkmark$$

$$\frac{\partial M}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \checkmark$$

Exact

Potential: $\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{y}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} \Rightarrow f(x,y) = \int \frac{x}{x^2+y^2}dy + g(x) = \arctan(y/x) + g($$

$$\int \frac{a}{a^2+y^2}dy = \int \frac{1/a}{1+(y/a)^2}dy = \arctan(y/a)$$

$$f(x,y) = \arctan\left(\frac{y}{x}\right) + g(x)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) + g'(x)$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} + g'(x) = -\frac{y}{x^2+y^2} + g'(x)$$

$$= \frac{1}{x} - \frac{y}{x^2+y^2}$$

$$\Rightarrow g'(x) = \frac{1}{x} \Rightarrow g(x) = \ln|x|$$

$$\boxed{f(x,y) = \arctan(y/x) + \ln|x|}$$

$\Rightarrow$ Solution to ODE: $\boxed{\arctan(y/x) + \ln|x| = C}$

Observation: What if, instead, we found the potential by integrating with respect to x first?

$$\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{y}{x^2+y^2} \Rightarrow f(x,y) = \ln|x| - \arctan(x/y) + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -\frac{1}{1+x^2/y^2}\cdot\frac{-x}{y^2} + g'(y)$$

$$= +\frac{x}{x^2+y^2} + g'(y)$$

$$= \frac{x}{x^2+y^2}$$

$$\Rightarrow f(x,y) = \ln|x| - \arctan(x/y) \Rightarrow \boxed{\ln|x| - \arctan(x/y) = C}$$

We obtained two apparently different solutions:

$$\boxed{\ln|x| + \arctan(y/x) = C} \quad (1)$$

$$\boxed{\ln|x| - \arctan(x/y) = C} \quad (2)$$

We can check by implicit differentiation that both satisfy the DE:

$$(1) \Rightarrow \frac{1}{x} + \frac{1}{1+(y/x)^2}\left(\frac{y'}{x} - \frac{y}{x^2}\right) = 0$$

$$\Rightarrow \frac{1}{x} + \frac{\cancel{x^2}}{x^2+y^2}\,\frac{xy'-y}{\cancel{x^2}} = 0 \Rightarrow \left(\frac{1}{x} - \frac{y}{x^2+y^2}\right) + \frac{x}{x^2+y^2}\,y' = 0 \quad ✓$$

$$(2) \Rightarrow \frac{1}{x} - \frac{1}{1+(x/y)^2}\left(\frac{1}{y} - \frac{x}{y^2}y'\right) = 0$$

$$\Rightarrow \frac{1}{x} - \frac{\cancel{y^2}}{x^2+y^2}\,\frac{y-xy'}{\cancel{y^2}} = 0 \Rightarrow \left(\frac{1}{x} - \frac{y}{x^2+y^2}\right) + \frac{x}{x^2+y^2}\,y' = 0 \quad ✓$$

The two solutions are equivalent, because

$$\boxed{\arctan\left(\frac{1}{\alpha}\right) = \pm\frac{\pi}{2} - \arctan(\alpha)}$$

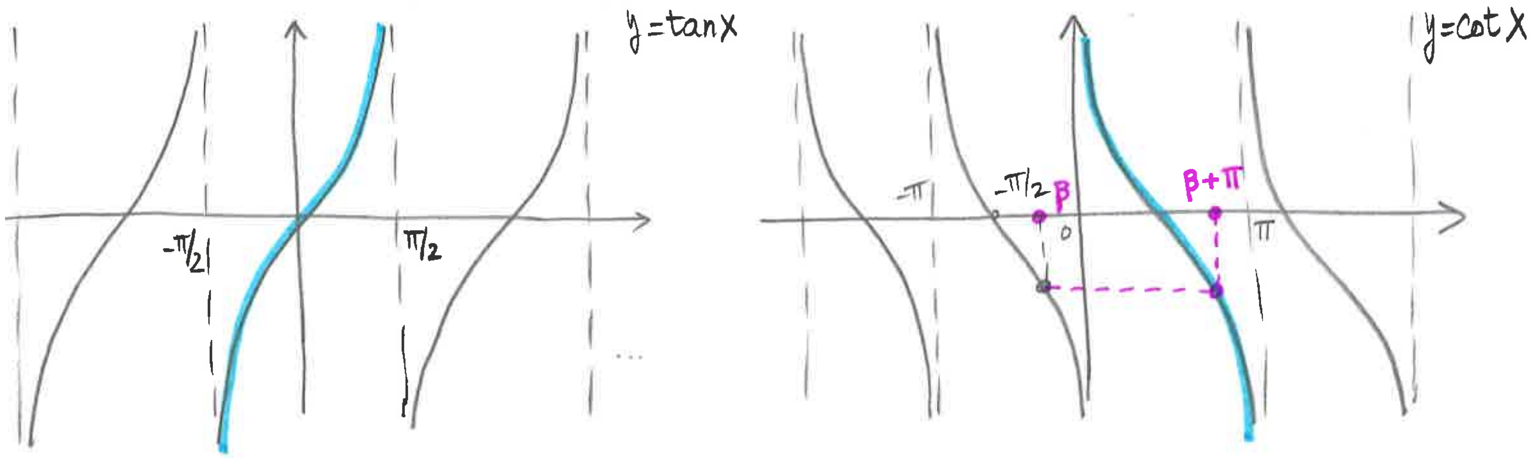
(proof on next page, only if you care)

So

$$(1): \ln|x| + \arctan(y/x) = C \iff \ln|x| \pm \frac{\pi}{2} - \arctan(x/y) = C$$

$$\iff \ln|x| - \arctan(x/y) = C \quad (2)$$

$$\boxed{\arctan\left(\frac{1}{\alpha}\right) = \begin{cases} -\frac{\pi}{2} - \arctan(\alpha), & \text{if } \alpha < 0 \\ \frac{\pi}{2} - \arctan(\alpha), & \text{if } \alpha > 0 \end{cases}}$$



Recall some basic facts from Trigonometry:

$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ $\qquad$ $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$

$$\boxed{\arctan(\alpha) + \operatorname{arccot}(\alpha) = \frac{\pi}{2}}$$

Suppose $\boxed{\arctan \frac{1}{\alpha} = \beta}$ $\hookrightarrow \beta \in (-\pi/2, \pi/2)$

$\Rightarrow \frac{1}{\alpha} = \tan\beta$

$\Rightarrow \alpha = \frac{1}{\tan\beta}$

$\Rightarrow \boxed{\alpha = \cot\beta} \Rightarrow \operatorname{arccot}(\alpha) = \operatorname{arccot}(\cot\beta)$

If $\boxed{\alpha > 0} \Rightarrow \boxed{\beta \in (0, \pi/2)} \Rightarrow \boxed{\operatorname{arccot}\alpha = \beta}$

If $\boxed{\alpha < 0} \Rightarrow \boxed{\beta \in (-\pi/2, 0)} \Rightarrow \boxed{\operatorname{arccot}\alpha = \beta + \pi}$ (see pic on the right above)

$$\Rightarrow \beta = \arctan\left(\frac{1}{\alpha}\right) = \begin{cases} \operatorname{arccot}\alpha & \text{if } \alpha > 0 \\ \operatorname{arccot}\alpha - \pi & \text{if } \alpha < 0 \end{cases} = \begin{cases} \frac{\pi}{2} - \arctan\alpha & \text{if } \alpha > 0 \\ -\frac{\pi}{2} - \arctan\alpha & \text{if } \alpha < 0 \end{cases}$$

13) $(x+y)^2 dx + (2xy + x^2 - 1) dy = 0 \;;\; y(1) = 1$

$$\frac{\partial M}{\partial y} = 2(x+y) \quad \checkmark$$

$$\frac{\partial N}{\partial x} = 2y + 2x = 2(x+y) \quad \checkmark$$

<u>Exact</u>

<u>Potential</u>: $\frac{\partial f}{\partial x} = (x+y)^2 \Rightarrow f(x,y) = \frac{1}{3}(x+y)^3 + g(y)$

$$\Rightarrow \left.\begin{aligned} \frac{\partial f}{\partial y} &= (x+y)^2 + g'(y) \\ &= 2xy + x^2 - 1 = (x+y)^2 - (1+y^2) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow g'(y) = -(1+y^2) \Rightarrow g(y) = -y - \frac{1}{3}y^3$$

$$\boxed{f(x,y) = \frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3}$$

$$\Rightarrow \boxed{\frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3 = c} \qquad \begin{matrix} x=1 \\ y=1 \end{matrix} \Rightarrow \frac{8}{3} - 1 - \frac{1}{3} = c \Rightarrow c = \frac{4}{3}$$

$$\Rightarrow \boxed{(x+y)^3 - 3y - y^3 = 4}$$

14) $(e^x + y) dx + (2 + x + ye^y) dy = 0 \;;\; y(0) = 1$

$$\frac{\partial M}{\partial y} = 1 \;;\; \frac{\partial N}{\partial x} = 1 \qquad \underline{\text{Exact}}$$

$$\frac{\partial f}{\partial x} = e^x + y \Rightarrow f(x,y) = e^x + xy + g(y)$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial f}{\partial y} &= x + g'(y) \\ &= 2 + x + ye^y \end{aligned}\right\} \begin{aligned} &\Rightarrow g'(y) = 2 + ye^y \\ &\Rightarrow g(y) = 2y + ye^y - e^y \end{aligned}$$

$$\boxed{f(x,y) = e^x + xy + 2y + ye^y - e^y} \Rightarrow \boxed{e^x + xy + 2y + ye^y - e^y = c}$$

$$x=0, y=1 \Rightarrow 1 + 2 + e - e = c \Rightarrow c = 3$$

$$\boxed{e^x + xy + 2y + ye^y - e^y = 3}$$

15. $(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + \ln y)dy = 0 \;;\; y(0) = e$

$$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2 \;;\; \frac{\partial N}{\partial x} = 2y\cos x - 3x^2 \qquad \underline{\text{Exact}}$$

$$\frac{\partial f}{\partial x} = y^2\cos x - 3x^2y - 2x \Rightarrow f(x,y) = y^2\sin x - x^3y - x^2 + g(y)$$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= 2y\sin x - x^3 + g'(y)\\ &= 2y\sin x - x^3 + \ln y\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow g'(y) = \ln y \Rightarrow g(y) = y\ln y - y$$

$$\boxed{f(x,y) = y^2\sin x - x^3y - x^2 + y\ln y - y} \Rightarrow \boxed{y^2\sin x - x^3y - x^2 + y\ln y - y = C}$$

$$x=0,\, y=e \Rightarrow e - e = C \Rightarrow C = 0$$

$$\Rightarrow \boxed{y^2\sin x - x^3y - x^2 + y\ln y - y = 0}$$

16. $\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x) \;;\; y(0) = 1$

$$y(y+\sin x)dx - \left(\frac{1}{1+y^2} + \cos x - 2xy\right)dy = 0$$

$$M(x,y) = y(y+\sin x) = y^2 + y\sin x \Rightarrow \frac{\partial M}{\partial y} = 2y + \sin x \;\checkmark$$

$$N(x,y) = -\left(\frac{1}{1+y^2} + \cos x - 2xy\right) \Rightarrow \frac{\partial N}{\partial x} = \sin x + 2y \;\checkmark \qquad \underline{\text{Exact}}$$

$$\frac{\partial f}{\partial x} = y^2 + y\sin x \Rightarrow f(x,y) = xy^2 - y\cos x + g(y)$$

$$\Rightarrow \left.\begin{aligned}\frac{\partial f}{\partial y} &= 2xy - \cos x + g'(y)\\ &= 2xy - \cos x - \frac{1}{1+y^2}\end{aligned}\right\} \Rightarrow g'(y) = -\frac{1}{1+y^2}$$

$$\Rightarrow g(y) = -\arctan(y)$$

$$\left.\begin{aligned}&\Rightarrow f(x,y) = xy^2 - y\cos x - \arctan(y) = C\\ &x=0,\, y=1 \Rightarrow \boxed{-1 - \frac{\pi}{4} = C}\end{aligned}\right\} \Rightarrow \boxed{xy^2 - y\cos x - \arctan(y) = -1 - \frac{\pi}{4}}$$

17 $(y^3+kxy^4-2x)\,dx+(3xy^2+20x^2y^3)\,dy=0$

$$\left.\begin{aligned} M(x,y) &= y^3+kxy^4-2x \Rightarrow \frac{\partial M}{\partial y} = 3y^2+4kxy^3 \\ N(x,y) &= 3xy^2+20x^2y^3 \Rightarrow \frac{\partial N}{\partial x} = 3y^2+40xy^3 \end{aligned}\right\} \Rightarrow 4k=40 \Rightarrow \boxed{k=10}$$

Solve: $\frac{\partial f}{\partial x} = y^3+10xy^4-2x \Rightarrow f(x,y)= xy^3+5x^2y^4-x^2+g(y)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= 3xy^2+20x^2y^3+g'(y) \\ &= 3xy^2+20x^2y^3 \end{aligned}\right\} \Rightarrow g'(y)=0 \Rightarrow g(y)=c$$

Solution: $\boxed{xy^3+5x^2y^4-x^2=C}$

18 $(2x-y\sin(xy)+ky^4)\,dx-(20xy^3+x\sin(xy))\,dy=0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\sin(xy)-xy\cos(xy)+4ky^3 \\ \frac{\partial N}{\partial x} &= -20y^3-\sin(xy)-xy\cos(xy) \end{aligned}\right\} \Rightarrow 4k=-20 \Rightarrow \boxed{k=-5}$$

Solve: $\frac{\partial f}{\partial x} = 2x-y\sin(xy)-5y^4 \Rightarrow f(x,y)= x^2+\cos(xy)-5y^4x+g(y)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= -x\sin(xy)-20y^3x+g'(y) \\ &= -x\sin(xy)-20y^3x \end{aligned}\right\} \Rightarrow g'(y)= \quad g(y)=$$

Solution: $\boxed{x^2+\cos(xy)-5y^4x=C}$

19) $(2xy^2 + ye^x)\,dx + (2x^2y + ke^x - 1)\,dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4xy + e^x \\ \frac{\partial N}{\partial x} &= 4xy + ke^x \end{aligned}\right\} \Rightarrow \boxed{k=1}$$

Solve: $\frac{\partial f}{\partial x} = 2xy^2 + ye^x \Rightarrow f(x,y) = x^2y^2 + ye^x + g(y)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= 2x^2y + e^x + g'(y) \\ &= 2x^2y + e^x - 1 \end{aligned}\right\} \Rightarrow g'(y) = -1 \Rightarrow g(y) = -y$$

$\Rightarrow$ Solution: $\boxed{x^2y^2 + ye^x - y = c}$

20) $(6xy^3 + \cos y)\,dx + (kx^2y^2 - x\sin y)\,dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 18xy^2 - \sin y \\ \frac{\partial N}{\partial x} &= 2kxy^2 - \sin y \end{aligned}\right\} \Rightarrow 18 = 2k \Rightarrow \boxed{k=9}$$

Solve: $\frac{\partial f}{\partial x} = 6xy^3 + \cos y \Rightarrow f(x,y) = 3x^2y^3 + x\cos y + g(y)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= 9x^2y^2 - x\sin y + g'(y) \\ &= 9x^2y^2 - x\sin y \end{aligned}\right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$$

$\Rightarrow$ Solution: $\boxed{3x^2y^3 + x\cos y = c}$

(21) $M(x,y)\,dx + \left(xe^{xy} + 2xy + \frac{1}{x}\right)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + 2y - \frac{1}{x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = e^{xy}(1+xy) + 2y - \frac{1}{x^2}$$

$$\Rightarrow \boxed{M(x,y) = ye^{xy} + y^2 - \frac{y}{x^2} + g(x)}$$

You can choose any function of x you want here

$$\int e^{xy}(1+xy)dy = \frac{1}{x}\int (1+xy)(e^{xy})'dy$$

$$= \frac{1}{x}(1+xy)e^{xy} - \frac{1}{x}\int xe^{xy}dy$$

$$= \left(\frac{1}{x}+y\right)e^{xy} - \frac{1}{x}e^{xy}$$

$$= ye^{xy}$$

(22) $\left(\frac{\sqrt{y}}{\sqrt{x}} + \frac{x}{x^2+y}\right)dx + N(x,y)\,dy = 0$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{2\sqrt{xy}} - \frac{x}{(x^2+y)^2}$$

$$\Rightarrow \boxed{N(x,y) = \frac{\sqrt{x}}{\sqrt{y}} + \frac{1}{2(x^2+y)} + g(y)}$$

this can be any function of y

23 $\underbrace{y(x+y+1)}_{M_1(x,y)}dx + \underbrace{(x+2y)}_{N_1(x,y)}dy = 0 \;;\; \mu(x,y) = e^x$

$(*)\quad ye^x(x+y+1)dx + (xe^x + 2ye^x)dy = 0$

$$\frac{\partial M}{\partial y} = e^x(x+2y+1) \quad ✓$$

$$\frac{\partial N}{\partial x} = e^x + xe^x + 2ye^x = e^x(x+2y+1) \quad ✓$$

Exact

Remark : The original equation is not exact : $\frac{\partial M_1}{\partial y} = x+2y+1$, $\frac{\partial N_1}{\partial x} = 1$

Solve $(*)$: $\frac{\partial f}{\partial x} = xye^x + y^2e^x + ye^x$

$\frac{\partial f}{\partial y} = xe^x + 2ye^x \Rightarrow f(x,y) = xye^x + y^2e^x + g(x)$

$\Rightarrow \frac{\partial f}{\partial x} = ye^x + xye^x + y^2e^x + g'(x) \Rightarrow g'(x) = 0$

$\Rightarrow$ Solution : $\boxed{xye^x + y^2e^x = C}$

24 $(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0 \;;\; \mu(x,y) = xy$

$(*)\quad (-x^2y^2\sin x + 2xy^2\cos x)dx + 2x^2y\cos x\,dy = 0$

$\frac{\partial M}{\partial y} = -2x^2y\sin x + 4xy\cos x \;;\; \frac{\partial N}{\partial x} = 4xy\cos x - 2x^2y\sin x \Rightarrow$ Exact

Solve : $\frac{\partial f}{\partial y} = 2x^2y\cos x \Rightarrow f(x,y) = x^2y^2\cos x + g(x)$

$$\left.\begin{aligned}\Rightarrow \frac{\partial f}{\partial x} &= 2xy^2\cos x - x^2y^2\sin x + g'(x)\\ &= 2xy^2\cos x - x^2y^2\sin x\end{aligned}\right\} \Rightarrow g'(x) = 0,\quad g(x) = C$$

$\Rightarrow$ Solution: $\boxed{x^2y^2\cos x = C}$