① $4y''+y'=0$

Char. Eqn.: $4m^2+m=0$

$m(4m+1)=0$

Roots: $0, -1/4$ $\Rightarrow$ Gen. Sol.: $\boxed{y=c_1+c_2e^{-x/4}}$

② $y''-36y=0$

Char. Eqn.: $m^2-36=0$

$(m-6)(m+6)=0$

Roots: $6,-6$ $\Rightarrow$ Gen. Sol.: $\boxed{y=c_1e^{6x}+c_2e^{-6x}}$

③ $12y''-5y'-2y=0$

Char. Eqn.: $12m^2-5m-2=0$

$\boxed{y=c_1e^{2x/3}+c_2e^{-x/4}}$

$\Delta=25+8\cdot12=121$

$$m=\frac{5\pm11}{24}=\frac{16}{24},\ \frac{-6}{24}$$

$$\frac{2}{3},\ \frac{-1}{4}$$

④ $y''-y=0;\ y(0)=y'(0)=1$

Char. Eqn.: $m^2-1=0$

$(m-1)(m+1)=0$

$m=\pm1$ $\Rightarrow$ gen. sol.: $y=c_1e^x+c_2e^{-x}$

$y'=c_1e^x-c_2e^{-x}$

$$\begin{cases} y(0)=1 \\ y'(0)=1 \end{cases} \qquad \begin{cases} c_1+c_2=1 \\ c_1-c_2=1 \end{cases}$$

$\oplus$ $2c_1=2 \Rightarrow c_1=1 \Rightarrow c_2=0 \Rightarrow$ $\boxed{\begin{array}{c} y=e^x \\ x\in\mathbb{R} \end{array}}$

⑤ $y'' - y = 0;\quad \underbrace{y(0)=1;\ y'(1)=0}_{\text{This is a } \underline{\underline{\text{BVP}}}}$

$y = c_1 e^{x} + c_2 e^{-x}$

$y' = c_1 e^{x} - c_2 e^{-x}$

(Same general sol. as above)

$$\begin{cases} y(0)=1 \\ y'(1)=0 \end{cases} \qquad \begin{cases} c_1 + c_2 = 1 \\ c_1 e - c_2 \frac{1}{e} = 0 \end{cases} \qquad \begin{cases} c_1 + c_2 = 1 \\ c_1 e^2 - c_2 = 0 \end{cases}$$

$c_2 = c_1 e^2$

$c_1 + c_1 e^2 = 1 \Rightarrow c_1 = \frac{1}{1+e^2}$

$\Rightarrow c_2 = \frac{e^2}{1+e^2}$

$$\Rightarrow \boxed{y = \frac{1}{1+e^2} e^{x} + \frac{e^2}{1+e^2} e^{-x}}$$

//

6) Find a homogeneous linear ODE which could have the solution:

$$y = 4e^{6x} + \pi e^{-2x}$$

Hint: look for an ODE with constant coefficients.

$$\underline{\underline{4}}e^{6x}, \quad \underline{\underline{\pi}}e^{-2x}$$

the coefficients 4 & π don't really matter here.
What does matter: the roots of the char. eqn., which would have to be $\underline{\underline{6}}$ and $\underline{\underline{-2}}$:

$$(m-6)(m+2)=0$$

$$m^2+2m-6m-12=0$$

$$m^2-4m-12=0 \longleftarrow \text{Char. Eqn.}$$

$\Rightarrow$ ODE: $\boxed{y''-4y'-12y=0}$

(7) Given that $y_1 = x^3$ is a solution of:

$$x^2 y'' - 6y = 0$$

use reduction of order to find a <u>second solution</u> on the interval $(0, \infty)$.

- Set $y = u(x)\,x^3$. Find the condition that $u(x)$ would have to satisfy so that $y = u \cdot x^3$ is a solution:

$$y = u \cdot x^3$$
$$y' = u' \cdot x^3 + 3x^2 \cdot u$$
$$y'' = u'' x^3 + u' \cdot 3x^2 + 6x \cdot u + 3x^2 \cdot u'$$
$$= u'' x^3 + u' \cdot 6x^2 + u \cdot 6x$$

$\Rightarrow$ $x^2 y'' - 6y = 0$ becomes:

$$x^2\left(x^3 u'' + 6x^2 u' + 6xu\right) - 6\left(x^3 u\right) = 0$$

$$x^5 u'' + 6x^4 u' + \cancel{6x^3 u} - \cancel{6x^3 u} = 0$$

$$x^4\left(x u'' + 6u'\right) = 0$$

$\Rightarrow$ Condition on u: $\boxed{x u'' + 6u' = 0}$

- Make the substitution: $v = u'$
  $v' = u''$

$\Rightarrow$ condition on u becomes:

$$x v' + 6v = 0$$

$$x\frac{dv}{dx}+6v=0$$

$$x\frac{dv}{dx}=-6v$$

$$-\frac{1}{6v}dv=\frac{1}{x}dx$$

$$\frac{-1}{6}\ln|v|=\ln|x|+C$$
$$=\ln x+C \quad \text{b/c } x\in(0,\infty)$$

$$\ln|v|=-6\ln x+C$$

$$v=x^{-6}\cdot e^{C}$$

$$\Rightarrow \boxed{v=\frac{C}{x^6}} \Rightarrow u'=\frac{C}{x^6}$$

$$u=\frac{-C}{5x^5}+C'$$

$y_1=x^3$
$y_2=ux^3$

Take $\boxed{u=\frac{1}{x^5}}$ => New solution:

$$y_2=\frac{1}{x^5}\cdot x^3=\frac{1}{x^2}$$

General Solution: $y=c_1y_1+c_2y_2$

$$\boxed{y=c_1x^3+\frac{c_2}{x^2}}$$