① $y'' - 4y' + 5y = 0$

Char. Eqn.: $m^2 - 4m + 5 = 0$

$\Delta = 16 - 20 = -4 < 0$

$m = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$y = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x).$

② $3y'' + 2y' + y = 0$

Char. Eqn.: $3m^2 + 2m + 1 = 0$

$\Delta = 4 - 12 = -8$

$m = \frac{-2 \pm \sqrt{-8}}{6} = \frac{-2 \pm 2\sqrt{2}i}{6} = \frac{-1 \pm \sqrt{2}i}{3}$

$= \frac{-1}{3} \pm i \frac{\sqrt{2}}{3}$

$y = c_1 e^{-x/3} \cos\left(\frac{\sqrt{2}}{3}x\right) + c_2 e^{-x/3} \sin\left(\frac{\sqrt{2}}{3}x\right)$

③ $y'' - 4y = 0$

Char. Eqn.: $m^2 - 4 = 0$

$m = \pm 2$

$y = c_1 e^{2x} + c_2 e^{-2x}.$

④ $y'' + 4y = 0$

Char. Eqn.: $m^2 + 4 = 0$

$m^2 = -4$; $m = \pm 2i = \underline{0} \pm 2i$

$\hookrightarrow e^{\alpha x} = e^{0 \cdot x} = 1.$

$y = c_1 \cos(2x) + c_2 \sin(2x)$

(5) $y'' + y = 0; \quad y(\pi/3) = 0, \quad y'(\pi/3) = 2.$

Char. Eqn.: $m^2 + 1 = 0$

$m = \pm i$

General Solution:

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$y(\pi/3)$?

$$y' = -c_1 \sin(x) + c_2 \cos(x)$$

$y'(\pi/3)$?

$$y(\pi/3) = c_1 \cdot \frac{1}{2} + c_2 \cdot \frac{\sqrt{3}}{2} = 0$$

$$y'(\pi/3) = -c_1 \cdot \frac{\sqrt{3}}{2} + c_2 \cdot \frac{1}{2} = 2$$

$$\begin{cases} c_1 + \sqrt{3} c_2 = 0 \\ -\sqrt{3} c_1 + c_2 = 4 \end{cases}$$

$$\begin{array}{r} \sqrt{3} c_1 + 3 c_2 = 0 \\ -\sqrt{3} c_1 + c_2 = 4 \\ \hline + \quad 4 c_2 = 4 \end{array}$$

$c_2 = 1$

$\Rightarrow \quad -\sqrt{3} c_1 = 3 \Rightarrow c_1 = -\sqrt{3}$

$$\boxed{y = -\sqrt{3} \cos(x) + \sin(x) \quad x \in \mathbb{R}}$$

IVP Solution

⑥ $y'' - 10y' + 25y = 0;\ \underbrace{y(0)=1;\ y(1)=0}_{\text{BVP}}$

Char. Eqn.: $m^2 - 10m + 25 = 0$

$(m-5)^2 = 0 \Rightarrow m_1 = m_2 = 5$ Repeated root

$\Rightarrow$ general solution: $\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$

$y(0) = c_1 = 1$

$y(1) = c_1 e^5 + c_2 e^5 = 0$

$e^5 + c_2 e^5 = 0 \Rightarrow c_2 = -1$

$\Rightarrow$ BVP Solution: $\boxed{y = e^{5x} - x e^{5x}}$

⑦ $y'' + 4y = 0;\ \underbrace{y(0)=0;\ y(\pi)=0.}_{\text{BVP}}$

Char. Eqn.: $m^2 + 4 = 0$

$m = \pm 2i$

$\Rightarrow$ Gen. Sol.: $\boxed{y = c_1 \cos(2x) + c_2 \sin(2x).}$

$\left.\begin{array}{l} y(0) = c_1 = 0 \\ y(\pi) = c_1 = 0 \end{array}\right\} \Rightarrow c_1 = 0$ but c_2 can be anything!

$\Rightarrow$ BVP Solution: $\boxed{y = c \sin(2x)}$

Note that this is an infinite family of solutions (not unique), which can happen with linear BVP's, but not IVP's.