Method of Undetermined Coefficients

$$M^2 + 3M + 2 = 0$$

$$m^2 + 3w + 2 = 0$$

 $(w+1)(w+2) = 0 \Rightarrow w = -1, -2$ $y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$\Rightarrow$$
 2A = 6 => A=3 $y_p = 3$

General Solution:
$$y=c_1e^{-x}+c_2e^{-2x}+3$$

$$m^2+4=0 \Rightarrow m=\pm 2i$$

$$m^2+4=0 \Rightarrow m=\pm 2i$$
 $y_c=C_1\sin(2x)+C_2\cos(2x)$

Particular Solution: Initial guess would be yp = A som(2x) + Bcos(2x)

but these are contained in the complementary solution!

$$y_{p}' = A m w(2x) + 2A x cos(2x) + B cos(2x) - 2B x sin(2x)$$

$$y_p'' = 2A\cos(2x) + 2A\cos(2x) - 4Ax mm(2x) - 2B mm(2x)$$

$$=4A\cos(2x)-4Bmin(2x)-4Axmin(2x)-4Bx\cos(2x)$$

=>
$$y_p'' + 4y_p = 4A\cos(2x) - 4Bmn(2x) + (4A-4A) \times mn(2x) + (4B-4B) \times coa(2x)$$

= $3 mn(2x)$

=>
$$\begin{cases} 4A = 0 \\ -4B = 3 \end{cases}$$
 => $A = 0$; $B = -\frac{3}{4}$ => $\begin{cases} y_p = -\frac{3}{4} \times \cos(2x) \end{cases}$

General Solution:
$$y = C_1 \sin(2x) + C_2 \cos(2x) - \frac{3}{4} \times \cos(2x)$$

(3) y"+2y'+y= 51mx+3004(2x)

Complementary salution:
$$y''+2y'+y=0$$

 $w^2+2w+1=0$
 $(w+1)^2=0$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$y_p' = A\cos x - B\cos x + 2\cos (2x) - 2D\cos (2x)$$

$$y_p'' = -Ammx - Bcos x - 4c min(2x) - 4D cos (2x)$$

$$y_p'' + 2y_p' + y_p = -A mm x - B cosx - 4c mm(2x) - 4D cos(2x)$$

$$= M \times X + 3 \cos(2x)$$

$$\begin{cases}
-28 = 1 & \Rightarrow B = -1/2 \\
2A = 0 & \Rightarrow A = 0 \\
-3C - 4D = 0
\end{cases}$$

$$y_p = -\frac{1}{2}\cos x + \frac{12}{25}\sin (2x) - \frac{9}{25}\cos(2x)$$

$$-12C - 16D = 0$$

$$12C - 9D = 9$$

$$/ -25D = 9$$

$$D = -9/25$$

$$C = \frac{3}{4}(D+1) = \frac{3}{4} \cdot \frac{16}{25} = \frac{12}{25}$$

$$y = C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2} coax + \frac{12}{25} mn(2x) - \frac{9}{25} coa(2x)$$

$$100^2 - 1000 + 25 = 0$$

$$(W-5)^2=0$$

Particular Salution:
$$y_p = Ax + B$$

$$y_p' = A$$

$$y_p'' - 10y_p' + 25y_p = -10A + 25Ax + 25B$$

= 25Ax + (25B-10A) = 30 X+3

$$\begin{cases} 25A = 30 \\ 25B - 40A = 3 \end{cases}$$

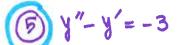
$$A = 6/5$$

$$B = \frac{1}{25}(10A+3) = \frac{1}{25} \cdot 15 \quad B = \frac{3}{5}$$

$$y_p = \frac{6}{5} \times + \frac{2}{5}$$

1/c = C1e5x+C2Xe5X

General Salution:
$$y = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}$$



$$w^2 - w = 0 \Rightarrow w(w-1) = 0 \Rightarrow w = 0,1$$

Particular Solution:
$$y_p = Ax$$
 (the smittal guess $y_p = A$ is contained in $y_c!$)
$$y'_p = A ; y''_p = 0$$

=>
$$y_p'' - y_p' = -A$$
 == A=3 => $y_p = 3x$

General Salution:
$$y = C_1 + C_2 e^{x} + 3x$$

$$6 \frac{1}{4}y'' + y' + y = x^2 - 2x$$

Complementary Solution:
$$\frac{1}{4}y'' + y' + y = 0$$

 $\frac{1}{4}w^2 + w + 1 = 0$
 $w^2 + 4w + 4 = 0$
 $(w + 2)^2 = 0$

$$y_c = c_1 e^{-2X} + c_2 x e^{-2X}$$

Particular Solution:
$$y_p = Ax^2 + Bx + C$$

 $y_p' = 2Ax + B$
 $y_p'' = 2A$

$$\frac{1}{4}y''_{P} + y'_{P} + y'_{P} = \frac{1}{2}A + 2AX + B + AX^{2} + BX + C$$

$$= AX^{2} + (2A + B)X + (\frac{1}{2}A + B + C) = X^{2} - 2X$$

$$\Rightarrow \begin{cases} A=1 \\ 2A+B=-2 \Rightarrow 2+B=-2 \Rightarrow B=-4 \\ \frac{1}{2}A+B+C=0 \Rightarrow \frac{1}{2}-4+C=0 \Rightarrow C=\frac{7}{2} \end{cases}$$

$$y_{p} = x^{2}-4x+\frac{7}{2}$$

$$y_{p} = \chi^{2} - 4\chi + \frac{7}{2}$$

$$(3) y'' + 3y = -48x^2 e^{3x}$$

Complementary Salution:
$$y''+3y=0$$

 $w^2+3=0 \Rightarrow m=\pm i\sqrt{3}$

Yc=C15/2(13x)+C2cos(13x

Particular Solution:
$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$y'_{\theta} = (2AX + B)e^{3X} + 3(AX^{2} + BX + C)e^{3X}
= (3AX^{2} + (3B + 2A)X + (B + 3C))e^{3X}
y''_{\theta} = (4AX + (3B + 3A)) + (3AX^{2} + (3A)) +$$

$$y_p'' = (6Ax + (3B+2A) + 9Ax^2 + (9B+6A)x + (3B+9C))e^{3x}$$

$$= (9Ax^2 + (9B+12A)x + (6B+2A+9C))e^{3x}$$

$$y_p'' + 3y_p = (12Ax^2 + (12B + 12A)x + (6B + 2A + 12C))e^{3x} = -48x^2e^{3x}$$

$$\Rightarrow \begin{cases} 12A = -48 & \Rightarrow \\ 12B + 12A = 0 \\ 6B + 2A + 12C = 0 \end{cases} \begin{cases} A = -4 \\ B = -A \Rightarrow B = 4 \end{cases} \qquad \begin{cases} y_P = (-4x^2 + 4x - 4/3)e^{3x} \\ 12C = -24 + 8 = -16 \Rightarrow C = -4/3 \end{cases}$$

General Solution:
$$y = C_1 \sin(\sqrt{3}x) + C_2 \cos(\sqrt{3}x) - (4x^2 + 4x + 4/3)e^{3x}$$

Particular Salution: Initial guess would be $y_p = (Ax+B)mmx + (Cx+D)cosx$ but this duplicates the complementary rodution. So, we multiply by x (and duplication is eliminated):

$$y_p = (Ax^2 + Bx) \sin x + (Cx^2 + Dx) \cos x$$

$$y_p' = (2Ax+B) mwx + (Ax^2+Bx) cosx$$

- $(Cx^2+Dx) mwx + (2Cx+D) cosx$

=
$$(-CX^2 + (2A-D)X+B)$$
mhx + $(AX^2 + (B+2C)X+D)$ cox X

$$J_{p}'' = (-Cx^{2} + (2A - D)x + B)\cos x + (-2Cx + (2A - D))mmx + (2Ax + (B+2C))\cos x - (Ax^{2} + (B+2C)x + D)mmx$$

$$= (-CX^{2} + (4A-D)X + (2B+2C))\cos X + (-AX^{2} - (B+4C)X + (2A-2D))s$$

=>
$$y_p'' + y_p = (-4Cx + (2A-2D)) = hux + (4Ax + (2B+2C)) cos x$$

= $2x mux$

$$C = -\frac{1}{2}$$
 $A = 0$
 $D = 0$
 $B = \frac{1}{2}$

 $y_p = \frac{1}{2} \times m \times -\frac{1}{2} \times \cos x$

General Solution:

y = C1 mmx + C2 cosx + 1 xmmx - 1 x2 cosx

Complementary Solution:
$$y''-y'+\frac{1}{4}y=0$$

 $w^2-w+\frac{1}{4}=0$
 $(w-1/2)^2=0$

$$y_c = c_1 e^{X/2} + c_2 x e^{X/2}$$

Particular Salution: Smitial guess would be A+Be^{N2}

$$y_p = A + B x^2 e^{x/2}$$

In this part reproduces the y_c , so multiply by $\frac{\chi^2}{d}$ (eliminates duplication

$$y'_{p} = 2BX e^{X/2} + \frac{1}{2}BX^{2}e^{X/2}$$

$$y''_{p} = 2Be^{X/2} + BX e^{X/2}$$

$$+ BX e^{X/2} + \frac{1}{4}BX^{2}e^{X/2}$$

$$y''_{P} - y'_{P} + \frac{1}{4}y_{P} =$$

$$(2B + 2BX + \frac{1}{4}BX^{2})e^{X/2}$$

$$- (2BX + \frac{1}{2}BX^{2})e^{X/2}$$

$$+ \frac{1}{4}A + \frac{1}{4}BX^{2}e^{X/2}$$

$$= \frac{1}{4}A + 2Be^{X/2}$$

$$= 3 + e^{X/2}$$

$$\Rightarrow \begin{cases} \frac{1}{4}A = 3 & \begin{cases} A = 12 \\ B = 1/2 \end{cases}$$

$$y_p = 12 + \frac{1}{2}x^2 e^{x/2}$$

Q: Why didn't we also multiply the A by X2?

In problem (8) we had g(x) = 2x mx and so we multiplied the initial guess [(Ax+B) 5 mx + (Cx+D) to by X' ~> both terms were multiplied.

So why not here? Because here (A) and (BeX12) are actually 2 different guesses for two

one actually 2 different guesses for two particular solutions - we are really doing Superposition here!

$$#9$$
: $g(x) = 3 + e^{x/2}$ (roum of functions)

guess: A guess: $Be^{X/2} | *x^2$ to avoid

#8):
$$g(x) = 2x \text{ mix}$$
 (one function)

gues: $(Ax+B) \text{ mix} + (Cx+D) \text{ cos} x$ · X

this does have two terms, but they both come from gCX

If in #8 g(x) were instead g(x) = $2x m x + e^{3x}$ we would leave the guess for e^{3x} alone, i.e. we would put $y_p = (Ax^2 + Bx) m x + (Cx^2 + Dx) cos x + E =$

Complementary Solution:
$$y''-2y'+5y=0$$

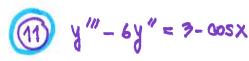
 $m^2-2m+5=0$
 $\Delta = 4-20=-16 \Rightarrow m = \frac{2\pm 4i}{2} = 1\pm 2i$
 $y_c = e^{\times}(c_1 \ln m(2x) + c_2 \cos(2x))$

Particular Salution: Initial guess would be: Aexcos(2x) + Bex 5m (2x)

(multiply by x to eliminate duplication of ye):

$$\begin{split} y_{p} &= Axe^{x} cos(2x) + Bxe^{x} sin(2x) \\ y_{p}' &= Ae^{x} cos(2x) + Axe^{x} cos(2x) - 2Axe^{x} sin(2x) \\ &+ Be^{x} sin(2x) + Bxe^{x} sin(2x) + 2Bxe^{x} cos(2x) \\ &= \left[\left(A + (A + 2B)x \right) cos(2x) + \left(B + (B - 2A)x \right) sin(2x) \right] e^{x} \\ y_{p}'' &= \left[\left(A + (A + 2B)x \right) cos(2x) + \left(B + (B - 2A)x \right) sin(2x) + \left(A + 2B \right) cos(2x) - 2\left(A + (A + 2B)x \right) sin(2x) + \left(B - 2A \right) sin(2x) + 2\left(B + (B - 2A)x \right) cos(2x) \right] e^{x} \\ &= \left[\left(2A + 4B + \left(4B - 3A \right)x \right) cos(2x) + \left(2B - 4A + \left(-4A - 3B \right)x \right) sin(2x) \right] e^{x} \\ &= e^{x} cos(2x) \\ \Rightarrow \begin{cases} 4B = 1 \\ -4A = 0 \end{cases} \begin{cases} B = \sqrt{4} \\ A = 0 \end{cases} \qquad \begin{cases} y_{p} &= \frac{4}{4} \times e^{x} sin(2x) \end{cases} \end{split}$$

General Solution:
$$y = c_1 e^{x} nnu(2x) + c_2 e^{x} cos(2x) + \frac{1}{4} x e^{x} snu(2x)$$



$$m^3 - 6m^2 = 0$$

 $m^2(m-6) = 0$

$$y_c = C_1 + C_2 X + C_3 e^{6X}$$

Particular Solution

these come from cosx

> this comes from the constant 3, Initial guess would be just "A" but we must multiply by X2 to eliminate deplication of the C, and C2X inyo.

$$y_{p} = Ax^{2} + B\cos x + C\cos x$$

$$y_{p}' = 2Ax - B\cos x + C\cos x$$

$$y_p'' = 2A - B \cos x - C \cos x$$

$$y_p''' = B m m x - C cos x$$

$$y_p''' - 6y_p'' = -12A + (6B-C) \cos x + (6C+B) \sin x$$

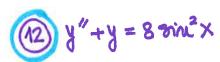
= 3-\coAx

$$\begin{cases}
-12A = 3 \\
6B-C = -1 \\
.6C+B = 0
\end{cases}$$

$$y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$$

$$C = 6B + 1$$

 $36B + 6 + B = 0$ $B = -6/37$
 $C = 1/37$



Complementary Solution: \ yc = C, mx + C2 cox X

Particular Solution: How to handle 881/2x? Trigonometric Identity:

 $\cos(2x) = \cos^2 x - 8m^2 x = 2\cos^2 x - 1 = 1 - 28m^2 x$ $\Rightarrow \sin^2 \lambda = \frac{1 - \cos(2\lambda)}{2}$

y'' + y = 4 - 4005(2X)

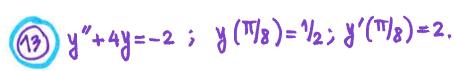
 $y_p' = -2B m (2x) + 2C cos (2x)$

 $y_p'' = -4B\cos(2x) - 4C \ln(2x)$

 $y_p = A + B \cos(2x) + C \sin(2x)$ $y_p'' + y_p = A - 3B \cos(2x) - 3C \sin(2x)$ =4-4008(2x)

 $y_p = 4 + \frac{4}{3}\cos(2x)$

y = C15MX + C2COSX + 4+ 4 COS(2X) General Solution:



Complementary Solution: y"+4y=0

 $y_c = C_1 \sin(2x) + C_2 \cos(2x)$

Particular Solution: $y_p = A \Rightarrow y_p' = y_p'' = 0$

 $\Rightarrow y_p'' + 4y_p = 4A$ $\Rightarrow A = 1/2$

General Solution: $y = C_1 \sin(2x) + C_2 \cos(2x) - 1/2$ y'= 20,00A(2X)-2020mm(2X)

IVP: y(T/8) = C1 2/+ C2/2-1/2=1/2 => (C1+C2) 1/2=1 => C1+C2=VZ y'(M8) = 2C1 = -2C2 = = 2 => (C1-C2) VZ=2 => C1-C2= 12

 $y = \sqrt{2} \ln(2x) - 1/2$

14) 5y"+y'=-6x; y(0)=0; y'(0)=-10

Complementary Solution: 5 m2+m=0

w(5m+1)=0 {0,-1/5} $y_c=c_1+c_2e^{-x/5}$

Particular Solution: $y_p = Ax^2 + Bx$ (initial guess Ax+B duplicates Cain yc) $y_p' = 2Ax + B$; $y_p' = 2A$

 $5y_p'' + y_p' = 10A + 2Ax + B$ $\begin{cases} 2A = -6 \\ 10A + B = 0 \end{cases}$

 $y_p = -3\chi^2 + 30X$

General Solution: $y = C_1 + C_2 e^{-x/5} - 3x^2 + 30x$ y'=-15c2e-x/5-6X+30

IVP: $0 = y(0) = C_1 + C_2$

 $-10=y'(0)=-\frac{1}{5}C_2+30$

C1+C2=0 $-\frac{1}{5}C_2 = -40$ $C_2 = 200$ $C_1 = -200$

Y = -200 + 200e -3X2+30X

(15) $y''+y=x^2+1$; y(0)=5, y(1)=0

Complementary Salution: Yc = C1 Situx+C2 cosx

Particular Salution: $y_P = Ax^2 + Bx + C$

 $y'_p = 2Ax + B$ $y''_p = 2A$

 $y_p'' + y_p = 2A + Ax^2 + Bx + C$ = $Ax^2 + Bx + (2A + C)$ = $x^2 + 1$

$$\begin{cases}
A=1 \\
B=0 \\
2A+C=1 \Rightarrow C=-1
\end{cases}$$

$$y_p = x^2 - 1$$

General Salution: $y = C_1 51 \ln X + C_2 \cos X + X^2 - 1$.

<u>IVP</u>: $5 = y(0) = C_2 - 1 \Rightarrow C_2 = 6$

$$0 = y(1) = c_1 \sin 1 + c_2 \cos 1 = c_1 \sin 1 = -6 \cos 1 = c_1 = -6 \cot ($$

 $y = -6\cot(1) \sin x + 6\cos x + x^2 - 1$