Method of Undetermined Coefficients

Homework 6 - Part 2 Solutions

1) $y'' + 3y' + 2y = 6$

Complementary Solution : $y'' + 3y' + 2y = 0$

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$$

Particular Solution : $y_p = A \Rightarrow y_p' = y_p'' = 0$

$\Rightarrow 2A = 6 \Rightarrow A = 3$ $\boxed{y_p = 3}$

General Solution : $\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + 3}$

2) $y'' + 4y = 3\sin(2x)$.

Complementary Solution : $y'' + 4y = 0$

$m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$\boxed{y_c = c_1 \sin(2x) + c_2 \cos(2x)}$$

Particular Solution : Initial guess would be $y_p = A\sin(2x) + B\cos(2x)$ but these are contained in the complementary solution!

So take $y_p = Ax\sin(2x) + Bx\cos(2x)$

$$y_p' = A\sin(2x) + 2Ax\cos(2x) + B\cos(2x) - 2Bx\sin(2x)$$

$$\begin{aligned} y_p'' &= 2A\cos(2x) + 2A\cos(2x) - 4Ax\sin(2x) - 2B\sin(2x) \\ &\quad - 2B\sin(2x) - 4Bx\cos(2x) \\ &= 4A\cos(2x) - 4B\sin(2x) - 4Ax\sin(2x) - 4Bx\cos(2x) \end{aligned}$$

$$\Rightarrow y_p'' + 4y_p = 4A\cos(2x) - 4B\sin(2x) + (\cancel{4A - 4A})x\sin(2x) + (\cancel{4B - 4B})x\cos(2x) = 3\sin(2x)$$

$$\Rightarrow \begin{cases} 4A = 0 \\ -4B = 3 \end{cases} \Rightarrow A = 0;\ B = -3/4 \Rightarrow \boxed{y_p = -\frac{3}{4}x\cos(2x)}$$

General Solution : $\boxed{y = c_1\sin(2x) + c_2\cos(2x) - \frac{3}{4}x\cos(2x)}$

③ $y'' + 2y' + y = \sin x + 3\cos(2x)$

Complementary solution: $y''+2y'+y=0$

$m^2+2m+1=0$

$(m+1)^2=0$

$m=-1$ (double root)

$$\boxed{y_c = c_1 e^{-x} + c_2 x e^{-x}}$$

Particular Solution: $y_p = A\sin x + B\cos x + C\sin(2x) + D\cos(2x)$

$y_p' = A\cos x - B\sin x + 2C\cos(2x) - 2D\sin(2x)$

$y_p'' = -A\sin x - B\cos x - 4C\sin(2x) - 4D\cos(2x)$

$$\begin{aligned} y_p'' + 2y_p' + y_p = &-A\sin x - B\cos x - 4C\sin(2x) - 4D\cos(2x) \\ &-2B\sin x + 2A\cos x - 4D\sin(2x) + 4C\cos(2x) \\ &+A\sin x + B\cos x + C\sin(2x) + D\cos(2x) \\ \hline &-2B\sin x + 2A\cos x + (-3C-4D)\sin(2x) + (4C-3D)\cos(2x) \\ &= \sin x + 3\cos(2x) \end{aligned}$$

$$\begin{cases} -2B = 1 & \Rightarrow B = -1/2 \\ 2A = 0 & \Rightarrow A = 0 \\ -3C-4D=0 \\ 4C-3D=3 \end{cases}$$

$$\boxed{y_p = -\frac{1}{2}\cos x + \frac{12}{25}\sin(2x) - \frac{9}{25}\cos(2x)}$$

$-12C-16D=0$

$12C-9D=9$

$/ \quad -25D=9 \qquad D=-9/25$

$C = \frac{3}{4}(D+1) = \frac{3}{4}\cdot\frac{16}{25} = \frac{12}{25}$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} - \frac{1}{2}\cos x + \frac{12}{25}\sin(2x) - \frac{9}{25}\cos(2x)}$$

4 $y'' - 10y' + 25y = 30x + 3$

Complementary Solution : $y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

$m = 5$ (double root)

$$\boxed{y_c = c_1 e^{5x} + c_2 x e^{5x}}$$

Particular Solution : $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$$y_p'' - 10y_p' + 25y_p = -10A + 25Ax + 25B$$

$$= 25Ax + (25B - 10A) = 30x + 3$$

$$\begin{cases} 25A = 30 \\ 25B - 10A = 3 \end{cases}$$

$A = 6/5$

$$B = \frac{1}{25}(10A + 3) = \frac{1}{25} \cdot 15 \qquad B = 3/5$$

$$\boxed{y_p = \frac{6}{5}x + \frac{3}{5}}$$

General Solution : $\boxed{y = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}}$

5 $y'' - y' = -3$

Complementary Solution : $y'' - y' = 0$

$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1$

$$\boxed{y_c = c_1 + c_2 e^x}$$

Particular Solution : $y_p = Ax$ (the initial guess $y_p = A$ is contained in y_c!)

$y_p' = A$; $y_p'' = 0$

$$\Rightarrow \left.\begin{aligned} y_p'' - y_p' &= -A \\ &= -3 \end{aligned}\right\} \Rightarrow A = 3 \Rightarrow \boxed{y_p = 3x}$$

General Solution : $\boxed{y = c_1 + c_2 e^x + 3x}$

⑥ $\frac{1}{4}y''+y'+y=x^2-2x$

Complementary Solution: $\frac{1}{4}y''+y'+y=0$

$$\frac{1}{4}m^2+m+1=0$$
$$m^2+4m+4=0$$
$$(m+2)^2=0$$

$$\boxed{y_c=c_1e^{-2x}+c_2xe^{-2x}}$$

Particular Solution: $y_p=Ax^2+Bx+C$

$$y_p'=2Ax+B$$
$$y_p''=2A$$

$$\frac{1}{4}y_p''+y_p'+y_p=\frac{1}{2}A+2Ax+B+Ax^2+Bx+C$$
$$=Ax^2+(2A+B)x+\left(\frac{1}{2}A+B+C\right)=x^2-2x$$

$$\Rightarrow\begin{cases} A=1 \\ 2A+B=-2 \Rightarrow 2+B=-2 \Rightarrow B=-4 \\ \frac{1}{2}A+B+C=0 \Rightarrow \frac{1}{2}-4+C=0 \Rightarrow C=7/2 \end{cases}$$

$$\boxed{y_p=x^2-4x+7/2}$$

General Solution: $\boxed{y=c_1e^{-2x}+c_2xe^{-2x}+x^2-4x+7/2}$

⑦ $y''+3y=-48x^2e^{3x}$

Complementary Solution: $y''+3y=0$

$$m^2+3=0 \Rightarrow m=\pm i\sqrt{3}$$

$$\boxed{y_c=c_1\sin(\sqrt{3}x)+c_2\cos(\sqrt{3}x)}$$

Particular Solution: $y_p=(Ax^2+Bx+C)e^{3x}$

$$y_p'=(2Ax+B)e^{3x}+3(Ax^2+Bx+C)e^{3x}$$
$$=(3Ax^2+(3B+2A)x+(B+3C))e^{3x}$$
$$y_p''=(6Ax+(3B+2A)+9Ax^2+(9B+6A)x+(3B+9C))e^{3x}$$
$$=(9Ax^2+(9B+12A)x+(6B+2A+9C))e^{3x}$$

$$y_p''+3y_p=(12Ax^2+(12B+12A)x+(6B+2A+12C))e^{3x}=-48x^2e^{3x}$$

$$\Rightarrow\begin{cases} 12A=-48 \\ 12B+12A=0 \\ 6B+2A+12C=0 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=-A \Rightarrow B=4 \\ 12C=-24+8=-16 \Rightarrow C=-4/3 \end{cases}$$

$$\boxed{y_p=(-4x^2+4x-4/3)e^{3x}}$$

General Solution: $\boxed{y=c_1\sin(\sqrt{3}x)+c_2\cos(\sqrt{3}x)-(4x^2-4x+4/3)e^{3x}}$

⑧ $y'' + y = 2x\sin x$

Complementary Solution: $y''+y=0$

$$m^2+1=0$$
$$m=\pm i$$

$$\boxed{y_c = C_1 \sin x + C_2 \cos x}$$

Particular Solution: Initial guess would be $y_p = (Ax+B)\sin x + (Cx+D)\cos x$ but this duplicates the complementary solution. So, we multiply by x (and duplication is eliminated):

$$y_p = (Ax^2+Bx)\sin x + (Cx^2+Dx)\cos x$$

$$\begin{aligned}
y_p' &= (2Ax+B)\sin x + (Ax^2+Bx)\cos x \\
&\quad - (Cx^2+Dx)\sin x + (2Cx+D)\cos x \\
&= (-Cx^2+(2A-D)x+B)\sin x + (Ax^2+(B+2C)x+D)\cos x \\
y_p'' &= (-Cx^2+(2A-D)x+B)\cos x + (-2Cx+(2A-D))\sin x \\
&\quad + (2Ax+(B+2C))\cos x - (Ax^2+(B+2C)x+D)\sin x \\
&= (-Cx^2+(4A-D)x+(2B+2C))\cos x + (-Ax^2-(B+4C)x+(2A-2D))\sin x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y_p''+y_p &= (-4Cx+(2A-2D))\sin x + (4Ax+(2B+2C))\cos x \\
&= 2x\sin x
\end{aligned}$$

$$\Rightarrow \begin{cases} -4C=2 \\ 2A-2D=0 \\ 4A=0 \\ 2B+2C=0 \end{cases} \qquad \begin{cases} C=-1/2 \\ A=0 \\ D=0 \\ B=1/2 \end{cases}$$

$$\boxed{y_p = \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x}$$

General Solution: $$\boxed{y = C_1\sin x + C_2\cos x + \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x}$$

9. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

Complementary Solution: $y'' - y' + \frac{1}{4}y = 0$

$$m^2 - m + \frac{1}{4} = 0$$
$$(m - 1/2)^2 = 0$$

$$\boxed{y_c = c_1 e^{x/2} + c_2 x e^{x/2}}$$

Particular Solution: Initial guess would be $A + \underbrace{Be^{x/2}}$

↳ this part reproduces the y_c, so multiply by $\underline{\underline{x^2}}$ (eliminates duplication)

$$y_p = A + Bx^2 e^{x/2}$$

$$y_p' = 2Bx\,e^{x/2} + \frac{1}{2}Bx^2 e^{x/2}$$

$$y_p'' = 2Be^{x/2} + Bx\,e^{x/2} + Bxe^{x/2} + \frac{1}{4}Bx^2 e^{x/2}$$

$$y_p'' - y_p' + \frac{1}{4}y_p =$$
$$\left(2B + 2Bx + \frac{1}{4}Bx^2\right)e^{x/2}$$
$$- \left(2Bx + \frac{1}{2}Bx^2\right)e^{x/2}$$
$$+ \frac{1}{4}A + \frac{1}{4}Bx^2 e^{x/2}$$

$$= \frac{1}{4}A + 2Be^{x/2}$$
$$= 3 + e^{x/2}$$

$$\Rightarrow \begin{cases} \frac{1}{4}A = 3 \\ 2B = 1 \end{cases} \quad \begin{cases} A = 12 \\ B = 1/2 \end{cases}$$

$$\boxed{y_p = 12 + \frac{1}{2}x^2 e^{x/2}}$$

Q: Why didn't we also multiply the A by x^2?

In problem 8 we had $g(x) = 2x\sin x$ and so we multiplied the initial guess $[(Ax+B)\sin x + (Cx+D)c$ by x ⤳ both terms were multiplied.

So why not here? Because here A and $Be^{x/2}$ are actually 2 different guesses for two particular solutions – we are really doing superposition here!

#9: $g(x) = 3 + e^{x/2}$ (sum of functions)

guess: A guess: $Be^{x/2}$ | $\cdot x^2$ to avoid'

#8: $g(x) = 2x\sin x$ (one function)

guess: $\underbrace{\left[(Ax+B)\sin x + (Cx+D)\cos x\right]}_{\text{this does have two terms, but they both come from } g(x}\cdot x$

If in #8 $g(x)$ were instead $g(x) = 2x\sin x + e^{3x}$ we would leave the guess for e^{3x} alone, i.e. we would put $y_p = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x + Ee^{3}$

General Solution: $$\boxed{y = c_1 e^{x/2} + c_2 x e^{x/2} + \frac{1}{2}x^2 e^{x/2} + 12}$$

(10) $y'' - 2y' + 5y = e^x \cos(2x)$

Complementary Solution : $y'' - 2y' + 5y = 0$

$$m^2 - 2m + 5 = 0$$

$$\Delta = 4 - 20 = -16 \Rightarrow m = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$\boxed{y_c = e^x \left(c_1 \sin(2x) + c_2 \cos(2x)\right)}$$

Particular Solution : Initial guess would be : $Ae^x \cos(2x) + Be^x \sin(2x)$
(multiply by x to eliminate duplication of y_c) :

$$y_p = Axe^x \cos(2x) + Bxe^x \sin(2x)$$

$$y_p' = Ae^x \cos(2x) + Axe^x \cos(2x) - 2Axe^x \sin(2x) + Be^x \sin(2x) + Bxe^x \sin(2x) + 2Bxe^x \cos(2x)$$

$$= \left[\left(A + (A+2B)x\right)\cos(2x) + \left(B + (B-2A)x\right)\sin(2x)\right] e^x$$

$$y_p'' = \left[\left(A + (A+2B)x\right)\cos(2x) + \left(B + (B-2A)x\right)\sin(2x) + (A+2B)\cos(2x) - 2\left(A + (A+2B)x\right)\sin(2x) + (B-2A)\sin(2x) + 2\left(B + (B-2A)x\right)\cos(2x)\right] e^x$$

$$= \left[\left(2A + 4B + (4B - 3A)x\right)\cos(2x) + \left(2B - 4A + (-4A - 3B)x\right)\sin(2x)\right] e$$

$$y_p'' - 2y_p' + 5y_p = (4B)\cos(2x)e^x + (-4A)\sin(2x)e^x$$

$$= e^x \cos(2x)$$

$$\Rightarrow \begin{cases} 4B = 1 \\ -4A = 0 \end{cases} \quad \begin{cases} B = 1/4 \\ A = 0 \end{cases} \qquad \boxed{y_p = \frac{1}{4} x e^x \sin(2x)}$$

General Solution : $\boxed{y = c_1 e^x \sin(2x) + c_2 e^x \cos(2x) + \frac{1}{4} x e^x \sin(2x)}$

11 $y''' - 6y'' = 3 - \cos x$

Complementary Solution : $y''' - 6y'' = 0$

$m^3 - 6m^2 = 0$

$m^2(m-6) = 0$

$$\boxed{y_c = C_1 + C_2 x + C_3 e^{6x}}$$

Particular Solution : $y_p = \underbrace{Ax^2}_{} + \underbrace{B\cos x + C\sin x}_{\text{these come from } \cos x}$

this comes from the constant 3. Initial guess would be just "A" but we must multiply by x^2 to eliminate duplication of the C_1 and $C_2 x$ in y_c.

$y_p = Ax^2 + B\cos x + C\sin x$

$y_p' = 2Ax - B\sin x + C\cos x$

$y_p'' = 2A - B\cos x - C\sin x$

$y_p''' = B\sin x - C\cos x$

$y_p''' - 6y_p'' = -12A + (6B - C)\cos x + (6C + B)\sin x$

$= 3 - \cos x$

$$\begin{cases} -12A = 3 \\ 6B - C = -1 \\ 6C + B = 0 \end{cases}$$

$A = -1/4$

$C = 6B + 1$

$36B + 6 + B = 0$ $\quad B = -6/37$

$\Rightarrow C = 1/37$

$$\boxed{y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

General Solution : $$\boxed{y = C_1 + C_2 x + C_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

(12) $y'' + y = 8\sin^2 x$

Complementary Solution : $\boxed{y_c = C_1 \sin x + C_2 \cos x}$

Particular Solution : How to handle $8\sin^2 x$? Trigonometric Identity :

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

$$\Rightarrow \boxed{\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}}$$

$$y'' + y = 4 - 4\cos(2x)$$

$$y_p = A + B\cos(2x) + C\sin(2x)$$

$$y_p' = -2B\sin(2x) + 2C\cos(2x)$$

$$y_p'' = -4B\cos(2x) - 4C\sin(2x)$$

$$y_p'' + y_p = A - 3B\cos(2x) - 3C\sin(2x)$$
$$= 4 - 4\cos(2x)$$

$$\begin{cases} A = 4 \\ 3B = 4 \Rightarrow B = 4/3 \\ 3C = 0 \Rightarrow C = 0 \end{cases}$$

$$\boxed{y_p = 4 + \frac{4}{3}\cos(2x)}$$

General Solution : $\boxed{y = C_1 \sin x + C_2 \cos x + 4 + \frac{4}{3}\cos(2x)}$

(13) $y''+4y=-2$; $y(\pi/8)=1/2$; $y'(\pi/8)=2$.

Complementary Solution : $y''+4y=0$
$m^2+4=0$
$m=\pm 2i$

$$\boxed{y_c = C_1 \sin(2X) + C_2\cos(2X)}$$

Particular Solution : $y_p = A \Rightarrow y_p' = y_p'' = 0$
$\Rightarrow y_p''+4y_p = 4A = -2 \Rightarrow A=-1/2$; $y_p=-1/2$

General Solution : $\boxed{y = C_1 \sin(2X) + C_2\cos(2X) - 1/2}$
$y' = 2C_1\cos(2X) - 2C_2 \sin(2X)$

IVP : $y(\pi/8) = C_1\frac{\sqrt{2}}{2} + C_2\frac{\sqrt{2}}{2} - 1/2 = 1/2 \Rightarrow (C_1+C_2)\frac{1}{\sqrt{2}} = 1 \Rightarrow C_1+C_2=\sqrt{2}$
$y'(\pi/8) = 2C_1\frac{\sqrt{2}}{2} - 2C_2\frac{\sqrt{2}}{2} = 2 \Rightarrow (C_1-C_2)\sqrt{2}=2 \Rightarrow C_1-C_2=\sqrt{2}$
$2C_1=2\sqrt{2}$
$C_1=\sqrt{2}$; $C_2=0$

$$\boxed{y=\sqrt{2}\sin(2X)-1/2}$$

(14) $5y''+y'=-6X$; $y(0)=0$; $y'(0)=-10$

Complementary Solution : $5m^2+m=0$
$m(5m+1)=0$ $\{0,-1/5\}$

$$\boxed{y_c = C_1 + C_2 e^{-X/5}}$$

Particular Solution : $y_p = AX^2+BX$ (initial guess $AX+B$ duplicates C_1 in y_c)
$y_p' = 2AX+B$; $y_p''=2A$
$5y_p''+y_p' = 10A+2AX+B = 2AX+(10A+B) = -6X$
$\begin{cases} 2A=-6 \\ 10A+B=0 \end{cases}$ $A=-3$; $B=30$

$$\boxed{y_p = -3X^2+30X}$$

General Solution : $\boxed{y = C_1 + C_2 e^{-X/5} - 3X^2 + 30X}$
$y' = -\frac{1}{5}C_2 e^{-X/5} - 6X + 30$

IVP : $0=y(0)=C_1+C_2$
$-10=y'(0) = -\frac{1}{5}C_2+30$

$C_1+C_2=0$
$-\frac{1}{5}C_2=-40$ $C_2=200$ $C_1=-200$

$$\boxed{y = -200+200e^{-X/5} - 3X^2+30X}$$

15 $y'' + y = x^2 + 1$; $y(0) = 5$, $y(1) = 0$

Complementary Solution : $\boxed{y_c = c_1 \sin x + c_2 \cos x}$

Particular Solution : $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$\begin{aligned} y_p'' + y_p &= 2A + Ax^2 + Bx + C \\ &= Ax^2 + Bx + (2A + C) \\ &= x^2 + 1 \end{aligned}$$

$$\begin{cases} A = 1 \\ B = 0 \\ 2A + C = 1 \Rightarrow C = -1 \end{cases}$$

$$\boxed{y_p = x^2 - 1}$$

General Solution : $y = c_1 \sin x + c_2 \cos x + x^2 - 1$.

IVP : $5 = y(0) = c_2 - 1 \Rightarrow c_2 = 6$

$0 = y(1) = c_1 \sin 1 + c_2 \cos 1 \Rightarrow c_1 \sin 1 = -6 \cos 1 \Rightarrow c_1 = -6\cot($

$$\boxed{y = -6\cot(1) \sin x + 6\cos x + x^2 - 1}$$