Variation of Parameters

① $y'' + 3y' + 2y = \frac{1}{1+e^x}$; $x \in \mathbb{R}$

Char. Eqn.: $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$\boxed{y_1 = e^{-x} ; \; y_2 = e^{-2x}}$

$f(x) = \frac{1}{1+e^x}$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{-2x}}{1+e^x} \; ; \; W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x}$$

$\Downarrow$

$\boxed{u_1 = \ln(1+e^x)}$

$$u_1' = \frac{-\frac{e^{-2x}}{1+e^x}}{-e^{-3x}} = \frac{e^{3x}}{e^{2x}(1+e^x)} = \frac{e^x}{1+e^x} \Rightarrow u_1 = \int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)$$

$\left(\text{u-sub: } u = 1+e^x, \; du = e^x dx \right)$

$$u_2' = \frac{\frac{e^{-x}}{1+e^x}}{-e^{-3x}} = -\frac{e^{3x}}{e^x(1+e^x)} = -\frac{e^{2x}}{e^x+1} \Rightarrow u_2 = -\int \frac{e^{2x}}{e^x+1}\,dx$$

$$= -\int \frac{e^x \cdot e^x}{e^x+1}\,dx = -\int \frac{e^x \cdot (e^x+1-1)}{e^x+1}\,dx$$

$$= -\int \frac{e^x(e^x+1) - e^x}{e^x+1}\,dx$$

$$= -\int \left(e^x - \frac{e^x}{e^x+1}\right) dx = -\left(e^x - \ln(e^x+1)\right)$$

$\boxed{u_2 = -e^x + \ln(e^x+1)}$

$$y_p = e^{-x}\ln(1+e^x) + e^{-2x}\left(-e^x + \ln(e^x+1)\right)$$

$$= \boxed{e^{-x}\ln(1+e^x) - e^{-x} + e^{-2x}\ln(e^x+1)}$$

General Solution: $y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1+e^x) - e^{-x} + e^{-2x}\ln(e^x+1)$.

② $y'' + 3y' + 2y = \sin(e^x)$; $x \in \mathbb{R}$

Complementary Solution: $y_c = C_1 e^{-x} + C_2 e^{-2x}$ (Same as #1)

$y_1 = e^{-x}$; $y_2 = e^{-2x}$

$W = -e^{-3x}$

$f(x) = \sin(e^x)$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin(e^x) & -2e^{-2x} \end{vmatrix} = -e^{-2x}\sin(e^x)$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin(e^x) \end{vmatrix} = e^{-x}\sin(e^x)$$

$$\Rightarrow u_1' = \frac{-e^{-2x}\sin(e^x)}{-e^{-3x}} = \frac{\sin(e^x)}{e^{2x}} \cdot e^{3x} = e^x \sin(e^x) \Rightarrow \boxed{u_1 = -\cos(e^x)}$$

$$\Rightarrow u_2' = \frac{e^{-x}\sin(e^x)}{-e^{-3x}} = -\frac{\sin(e^x)}{e^x} \cdot e^{3x} = -e^{2x}\sin(e^x)$$

$$u_2 = -\int e^{2x}\sin(e^x)\,dx$$

By Parts: $u = e^x$ $dv = -e^x \sin(e^x)dx$

$du = e^x dx$ $v = \cos(e^x)$

$$= \int e^x \cdot \left(-e^x \sin(e^x)\right)dx$$

$$= e^x \cos(e^x) - \int e^x \cos(e^x)\,dx$$

$$= e^x \cos(e^x) - \sin(e^x).$$

$$\boxed{u_2 = e^x \cos(e^x) - \sin(e^x)}$$

$$y_p = -\cos(e^x)e^{-x} + \left(e^x\cos(e^x) - \sin(e^x)\right)e^{-2x}$$

$$= \cancel{-\cos(e^x)e^{-x}} + \cancel{e^{-x}\cos(e^x)} - \sin(e^x)e^{-2x}$$

$$\boxed{y_p = -e^{-2x}\sin(e^x)}$$

$\Rightarrow$ General Solution: $\boxed{y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x}\sin(e^x)}$

③ $y'' + 3y' + 2y = e^{-x}$; $x \in \mathbb{R}$

$y_1 = e^{-x}$; $y_2 = e^{-2x}$; $W = -e^{-3x} = \frac{-1}{e^{3x}}$

$f(x) = e^{-x}$

$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x} \Rightarrow u_1' = \frac{-e^{-3x}}{-e^{-3x}} = 1 \Rightarrow \boxed{u_1 = x}$

$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \end{vmatrix} = e^{-2x} \Rightarrow u_2' = \frac{e^{-2x}}{-e^{-3x}} = -e^{x} \Rightarrow \boxed{u_2 = -e^{x}}$

$\Rightarrow y_p = e^{-x} \cdot x + e^{-2x}(-e^{x})$

$= e^{-x} \cdot x - e^{-x}$

$= (x-1)e^{-x}$

$\Rightarrow$ General Sol : $y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x}$

$y_p = (x-1)e^{-x}$

Enough to take $y_p = xe^{-x}$ b/c $-1 \cdot e^{-x}$ just duplicates y_c!

Note: This could have been solved via Undetermined Coefficients : $f(x) = e^{-x}$ is an exponential!

$y_c = c_1 e^{-x} + c_2 e^{-2x} \Rightarrow y_p = Axe^{-x}$ (to avoid duplication)

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$

$= -2Ae^{-x} + Axe^{-x}$

Replace in $y_p'' + 3y_p' + 2y_p = e^{-x}$:

$-2Ae^{-x} + \cancel{Axe^{-x}} + 3Ae^{-x} \cancel{-3Axe^{-x}} + \cancel{2Axe^{-x}} = e^{-x}$

$Ae^{-x} = e^{-x}$

$A = 1 \Rightarrow y_p = xe^{-x}$

④ $y'' + 2y' + y = e^{-x}\ln(x)\ ;\ x \in (0,\infty)$

Char. Eqn.: $m^2+2m+1=0$

$(m+1)^2 = 0 \Rightarrow m_{1,2} = -1$ $\qquad y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_1 = e^{-x}$

$y_2 = xe^{-x}$

$f(x) = e^{-x}\ln(x)$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}-xe^{-x} \end{vmatrix} = e^{-2x}\begin{vmatrix} 1 & x \\ -1 & 1-x \end{vmatrix} = e^{-2x}(1-x+x) = e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln(x) & e^{-x}(1-x) \end{vmatrix} = -x\ln(x)e^{-2x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln(x) \end{vmatrix} = e^{-2x}\ln(x)$$

$u_1' = -x\ln(x) \Rightarrow u_1 = -\int x\ln(x)\,dx$ $\qquad u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$= -\left(\frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx\right)$

$= -\frac{x^2}{2}\ln(x) + \frac{x^2}{4}$ $\qquad u_1 = \frac{-x^2}{2}\ln(x) + \frac{x^2}{4}$

$u_2' = \ln(x) \Rightarrow u_2 = \int \ln(x)\,dx$ $\qquad u = \ln(x) \quad dv = dx$

$= x\ln(x) - \int 1\,dx$ $\qquad du = \frac{1}{x}dx \quad v = x$

$= x\ln(x) - x$ $\qquad u_2 = x\ln(x) - x$

$\Rightarrow y_p = e^{-x}\left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}\right) + xe^{-x}\left(x\ln(x) - x\right)$

$= e^{-x}\left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4} + x^2\ln(x) - x^2\right)$

$= e^{-x}\left(\frac{x^2}{2}\ln(x) - \frac{3x^2}{4}\right)$

$y_p = e^{-x}\left(\frac{x^2}{2}\ln(x) - \frac{3x^2}{4}\right)$

$y = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}\left(\frac{x^2}{2}\ln(x) - \frac{3x^2}{4}\right)$ General Solution

⑤ $y'' - 2y' + y = \frac{e^x}{x}$; $x \in (0, \infty)$.

Complementary Solution: $m^2 - 2m + 1 = 0$
$(m-1)^2 = 0 \Rightarrow m_{1,2} = 1$ $y_c = c_1 e^x + c_2 x e^x$

$y_1 = e^x$, $y_2 = xe^x$

$f(x) = \frac{e^x}{x}$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = e^{2x}(x+1-x) = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x(1+x) \end{vmatrix} = -e^{2x} \Rightarrow u_1' = -1 \Rightarrow \boxed{u_1 = -x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x} \Rightarrow u_2' = \frac{1}{x} \Rightarrow \boxed{u_2 = \ln(x)} \text{ b/c } x > 0$$

$y_p = -xe^x + xe^x \ln(x) \Rightarrow$ General Solution: $\boxed{y = c_1 e^x + c_2 x e^x - xe^x + xe^x \ln(x)}$

⑥ Given that $y_1 = x$ and $y_2 = x\ln(x)$ form a fundamental set of solutions to

$$x^2y'' - xy' + y = 0$$

on $(0,\infty)$, find the general solution to

$$x^2y'' - xy' + y = 4x\ln(x) \quad ; \quad x \in (0,\infty).$$

CAUTION: STANDARD FORM !

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \underbrace{\frac{4\ln(x)}{x}}_{f(x)}$$

$$W = \begin{vmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{vmatrix} = x\ln(x) + x - x\ln(x) = x$$

$$W_1 = \begin{vmatrix} 0 & x\ln(x) \\ \frac{4\ln(x)}{x} & \ln(x)+1 \end{vmatrix} = -4\ln^2(x) \Rightarrow u_1' = -\frac{4\ln^2(x)}{x}$$

$$\Rightarrow u_1 = -4\int \frac{1}{x}\ln^2(x)\,dx = -4\cdot\frac{\ln^3(x)}{3}$$

$$u_1 = -\frac{4}{3}\ln^3(x)$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{4\ln(x)}{x} \end{vmatrix} = 4\ln(x) \Rightarrow u_2' = \frac{4\ln(x)}{x} \Rightarrow u_2 = 4\int \frac{1}{x}\ln(x)\,dx = 4\,\frac{\ln^2(x)}{2}$$

$$u_2 = 2\ln^2(x)$$

$$\Rightarrow y_p = -\frac{4}{3}x\ln^3(x) + 2x\ln^3(x) = \frac{2x}{3}\ln^3(x)$$

$$y_p = \frac{2x}{3}\ln^3(x)$$

General Solution: $\boxed{y = C_1x + C_2x\ln(x) + \frac{2x}{3}\ln^3(x)}$

⑦ Given that $y_1 = \frac{\cos(x)}{\sqrt{x}}$ and $y_2 = \frac{\sin(x)}{\sqrt{x}}$ form a fundamental set of solutions of

$$x^2 y'' + xy' + (x^2 - \tfrac{1}{4})y = 0$$

on $(0, \infty)$, find the general solution to

$$x^2 y'' + xy' + (x^2 - \tfrac{1}{4})y = x\sqrt{x} \quad , x \in (0,\infty).$$

Standard Form: $$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \frac{1}{\sqrt{x}} \longrightarrow f(x)$$

$$y_1 = \frac{\cos(x)}{\sqrt{x}} \; ; \; y_1' = \frac{-\sin(x)\sqrt{x} - \cos(x)\cdot\frac{1}{2\sqrt{x}}}{x} = \frac{-2x\sin(x) - \cos(x)}{2x\sqrt{x}}$$

$$y_2 = \frac{\sin(x)}{\sqrt{x}} \; ; \; y_2' = \frac{\cos(x)\sqrt{x} - \sin(x)\cdot\frac{1}{2\sqrt{x}}}{x} = \frac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$$

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{-2x\sin(x)-\cos(x)}{2x\sqrt{x}} & \frac{2x\cos(x)-\sin(x)}{2x\sqrt{x}} \end{vmatrix} = \left(2x\cos^2(x) - \cancel{\sin(x)\cos(x)} + 2x\sin^2(x) + \cancel{\sin(x)\cos(x)} \right)\frac{1}{2x^2}$$

$$= \frac{2x}{2x^2} = \frac{1}{x}$$

$$W_1 = \begin{vmatrix} 0 & \frac{\sin(x)}{\sqrt{x}} \\ \frac{1}{\sqrt{x}} & \frac{2x\cos(x)-\sin(x)}{2x\sqrt{x}} \end{vmatrix} = -\frac{\sin(x)}{x} \Rightarrow u_1' = -\sin(x) \Rightarrow \boxed{u_1 = \cos(x)}$$

$$W_2 = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & 0 \\ \frac{-2x\sin(x)-\cos(x)}{2x\sqrt{x}} & \frac{1}{\sqrt{x}} \end{vmatrix} = \frac{\cos(x)}{x} \Rightarrow u_2' = \cos(x) \Rightarrow \boxed{u_2 = \sin(x)}$$

$$\Rightarrow y_p = \frac{\cos^2(x)}{\sqrt{x}} + \frac{\sin^2(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}} \qquad \boxed{y_p = \frac{1}{\sqrt{x}}}$$

General Solution: $$\boxed{y = c_1 \frac{\cos(x)}{\sqrt{x}} + c_2 \frac{\sin(x)}{\sqrt{x}} + \frac{1}{\sqrt{x}}}$$