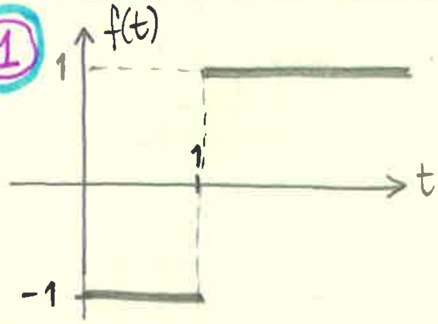
Laplace Transform: Definition

1



$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot(-1)\,dt + \int_1^\infty e^{-st}\cdot 1\,dt$$

$$= \frac{1}{s}e^{-st}\Big|_{t=0}^{1} - \frac{1}{s}e^{-st}\Big|_{t=1}^{\infty}$$

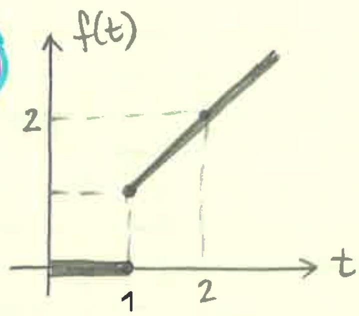
$$= \left(\frac{1}{s}e^{-s} - \frac{1}{s}\right) - \left(0 - \frac{1}{s}e^{-s}\right)$$

$[s>0]$

$$= \boxed{\frac{2}{s}e^{-s} - \frac{1}{s}}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s}e^{-s} - \frac{1}{s}\ ;\ s>0}$$

2



$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & 1 \le t < \infty \end{cases}$$

$$\int e^{-st}\cdot t\,dt = \int -\frac{1}{s}\left(e^{-st}\right)'\cdot t\,dt$$

$$= -\frac{t}{s}e^{-st} + \int \frac{1}{s}e^{-st}\,dt$$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

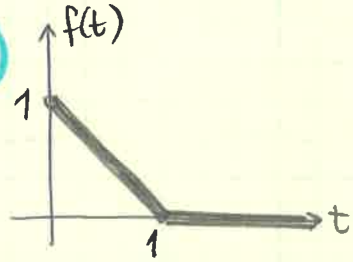
$$\mathcal{L}\{f(t)\} = \int_1^\infty e^{-st}\cdot t\,dt$$

$$= \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_{t=1}^{\infty}$$

$$= \frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-s} \qquad [s>0]$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{s+1}{s^2}e^{-s}\ ;\ s>0}$$

③



$$f(t)=\begin{cases}1-t\ , & 0\le t<1\\ 0\ , & t\geqslant 1\end{cases}$$

$$\mathcal{L}\{f(t)\}=\int_0^1(1-t)\,e^{-st}dt=\int_0^1 e^{-st}dt-\int_0^1 te^{-st}dt$$

$$=-\frac{1}{s}e^{-st}\Big|_{t=0}^{1}+\left(\frac{t}{s}e^{-st}+\frac{1}{s^2}e^{-st}\right)\Big|_{t=0}^{1}$$

$$=-\cancel{\frac{1}{s}e^{-s}}+\frac{1}{s}+\cancel{\frac{1}{s}e^{-s}}+\frac{1}{s^2}e^{-s}-\frac{1}{s^2}$$

$$\boxed{\mathcal{L}\{f(t)\}=\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-s}\quad ;\ s>0}$$

④ $f(t)=e^{t+7}$

$$\mathcal{L}\{e^{t+7}\}=\int_0^\infty e^{-st}e^{t+7}dt=e^7\int_0^\infty e^{(1-s)t}dt$$

$$=e^7\cdot\frac{1}{1-s}e^{(1-s)t}\Big|_{t=0}^{\infty}$$ converges iff $1-s<0$; $s>1$

$$=e^7\ \frac{-1}{1-s}\ \Rightarrow\ \boxed{\mathcal{L}\{e^{t+7}\}=\frac{e^7}{s-1}\ ;\ s>1}$$

⑤ $f(t)=te^{4t}$

$$\mathcal{L}\{te^{4t}\}=\int_0^\infty e^{-st}\cdot te^{4t}dt=\int_0^\infty te^{(4-s)t}dt$$

$$=\int_0^\infty t\cdot\frac{1}{4-s}\left(e^{(4-s)t}\right)'dt$$

Need $(4-s)<0$

$s>4$

$$=\frac{t}{4-s}e^{(4-s)t}\Big|_{t=0}^{\infty}-\int_0^\infty\frac{1}{4-s}e^{(4-s)t}dt$$

$$=0-\frac{1}{(4-s)^2}e^{(4-s)t}\Big|_0^\infty=\frac{1}{(4-s)^2}$$

$$\boxed{\mathcal{L}\{te^{4t}\}=\frac{1}{(4-s)^2}\ ;\ s>4}$$

⑥ $f(t) = t\cos t$

$$\mathcal{L}\{t\cos t\} = \int_0^\infty e^{-st}\cdot t\cos t\, dt$$

$$= \int_0^\infty -\frac{1}{s}(e^{-st})'\cdot t\cos t\, dt$$

$$= \underbrace{-\frac{1}{s}e^{-st}\cdot t\cos t\Big|_{t=0}^{\infty}}_{0} + \frac{1}{s}\int_0^\infty e^{-st}(\cos t - t\sin t)\, dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\cos t\, dt - \frac{1}{s}\int_0^\infty e^{-st}\cdot t\sin t\, dt$$

$$= \frac{1}{s}\mathcal{L}\{\cos t\} - \frac{1}{s}\mathcal{L}\{t\sin t\}$$

$$\boxed{\mathcal{L}\{t\cos t\} = \frac{1}{s}\mathcal{L}\{\cos t\} - \frac{1}{s}\mathcal{L}\{t\sin t\}} \qquad (1)$$

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st}\cos t\, dt$$

$$= \int_0^\infty -\frac{1}{s}(e^{-st})'\cos t\, dt$$

$$= \underbrace{-\frac{1}{s}e^{-st}\cos t\Big|_0^\infty}_{1/s} - \frac{1}{s}\int_0^\infty e^{-st}\sin t\, dt \qquad (s>0)$$

$$= \frac{1}{s} + \frac{1}{s^2}\int_0^\infty (e^{-st})'\sin t\, dt$$

$$= \frac{1}{s} + \underbrace{\frac{1}{s^2}e^{-st}\sin t\Big|_{t=0}^{\infty}}_{0} - \frac{1}{s^2}\underbrace{\int_0^\infty e^{-st}\cos t\, dt}_{\mathcal{L}\{\cos t\}}$$

$$\Rightarrow \mathcal{L}\{\cos t\} = \frac{1}{s} - \frac{1}{s^2}\mathcal{L}\{\cos t\} \; ; \; s>0$$

$$\Rightarrow \frac{s^2+1}{s^2}\mathcal{L}\{\cos t\} = \frac{1}{s} \Rightarrow \boxed{\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}} \qquad (2)$$

$$\mathcal{L}\{t\sin t\} = \int_0^\infty e^{-st}\, t\sin t\, dt$$

$$= \int_0^\infty \frac{-1}{s}\left(e^{-st}\right)' t\sin t\, dt$$

$$= \underbrace{\frac{-1}{s} e^{-st}\cdot t\sin t \Big|_{t=0}^{\infty}}_{0} + \int_0^\infty \frac{1}{s} e^{-st}(\sin t + t\cos t)\, dt$$

$$\Rightarrow \boxed{\mathcal{L}\{t\sin t\} = \frac{1}{s}\mathcal{L}\{\sin t\} + \frac{1}{s}\mathcal{L}\{t\cos t\}} \qquad (3)$$

$$\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st}\sin t\, dt = \int_0^\infty -\frac{1}{s}\left(e^{-st}\right)'\sin t\, dt$$

$$= \underbrace{-\frac{1}{s} e^{-st}\sin t \Big|_0^\infty}_{0} + \int_0^\infty \frac{1}{s} e^{-st}\cos t\, dt$$

$$= \frac{1}{s}\mathcal{L}\{\cos t\} = \frac{1}{s^2+1} \qquad (\text{from } (2))$$

$$\Rightarrow (3): \boxed{\mathcal{L}\{t\sin t\} = \frac{1}{s}\frac{1}{s^2+1} + \frac{1}{s}\mathcal{L}\{t\cos t\}} \qquad (4)$$

$\Rightarrow (1), (2) \,\&\, (4)$:

$$\mathcal{L}\{t\cos t\} = \frac{1}{s}\cdot\frac{s}{s^2+1} - \frac{1}{s}\left(\frac{1}{s}\frac{1}{s^2+1} + \frac{1}{s}\mathcal{L}\{t\cos t\}\right)$$

$$= \frac{1}{s^2+1} - \frac{1}{s^2(s^2+1)} - \frac{1}{s^2}\mathcal{L}\{t\cos t\}$$

$$\Rightarrow \frac{s^2+1}{s^2}\mathcal{L}\{t\cos t\} = \frac{s^2-1}{s^2(s^2+1)} \Rightarrow \boxed{\mathcal{L}\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2},\ s>0}$$

7 $f(t) = t\sin t$

From (4): $\mathcal{L}\{t\sin t\} = \dfrac{1}{s(s^2+1)} + \dfrac{s^2-1}{s(s^2+1)^2} = \dfrac{s^2+\cancel{1}+s^2-\cancel{1}}{s(s^2+1)^2}$

$$\boxed{\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}\ ;\ s>0}$$

8) $f(t) = e^{-t}\sin t$

$$\mathcal{L}\{e^{-t}\sin t\} = \int_0^\infty e^{-st}\cdot e^{-t}\sin t\,dt$$

$$= \int_0^\infty e^{-(s+1)t}\sin t\,dt$$

$$= \int_0^\infty -\frac{1}{s+1}e^{-(s+1)t}\sin t\,dt$$

$$= \underbrace{-\frac{1}{s+1}e^{-(s+1)t}\sin t\Big|_{t=0}^{\infty}}_{0 \text{ if } s+1>0;\ s>-1} + \int_0^\infty \frac{1}{s+1}e^{-(s+1)t}\cos t\,dt$$

$$= -\frac{1}{(s+1)^2}\int_0^\infty \left(e^{-(s+1)t}\right)'\cos t\,dt$$

$$= \underbrace{-\frac{1}{(s+1)^2}e^{-(s+1)t}\cos t\Big|_0^\infty}_{\frac{1}{(s+1)^2} \text{ if } s>-1} - \frac{1}{(s+1)^2}\int_0^\infty e^{-(s+1)t}\sin t\,dt$$

$$\Rightarrow \mathcal{L}\{e^{-t}\sin t\} = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}\mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow \frac{(s+1)^2+1}{(s+1)^2}\mathcal{L}\{e^{-t}\sin t\} = \frac{1}{(s+1)^2} \Rightarrow \boxed{\mathcal{L}\{e^{-t}\sin t\} = \frac{1}{s^2+2s+2};\quad s>-1}$$