Properties of the Laplace Transform

① $\mathcal{L}\{te^{10t}\} = -\frac{d}{ds}\mathcal{L}\{e^{10t}\} = -\frac{d}{ds}\left(\frac{1}{s-10}\right) = \frac{1}{(s-10)^2}$ $(s>10)$

$(s>10)$

② $\mathcal{L}\{t^3e^{-2t}\} = -\frac{d^3}{ds^3}\mathcal{L}\{e^{-2t}\}$ True, but it may be easier to use translation property

$= \mathcal{L}\{t^3\}\Big|_{s\to s+2}$

$= \frac{3!}{s^4}\Big|_{s\mapsto s+2} = \frac{6}{(s+2)^4}$; $(s>-2)$

$(s>0)\mapsto(s+2>0)$.

③ $\mathcal{L}\{e^t\sin(3t)\} = \mathcal{L}\{\sin(3t)\}_{s\mapsto s-1} = \frac{3}{s^2+9}\Big|_{s\mapsto s-1} = \frac{3}{(s-1)^2+9}$ $(s>1$

$(s>0)\mapsto(s-1>0)$

④ $\mathcal{L}\{e^{-t}\sin^2 t\} = \mathcal{L}\{\sin^2 t\}\Big|_{s\to s+1} = \frac{2}{(s+1)((s+1)^2+4)}$ $(s>-1)$

$(s>0)\mapsto(s+1>$

$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos(2t)}{2}\right\} = \mathcal{L}\{\tfrac{1}{2}\} - \frac{1}{2}\mathcal{L}\{\cos(2t)\}$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$

$(s>0)$ $(s>0)$ $(s>0)$

⑤ $\mathcal{L}\{t(e^t+e^{2t})^2\} = \mathcal{L}\{t(e^{2t}+2e^{3t}+e^{4t})\}$

$= \mathcal{L}\{te^{2t}\} + 2\mathcal{L}\{te^{3t}\} + \mathcal{L}\{te^{4t}\}$

$= \mathcal{L}\{t\}\Big|_{s\to s-2} + 2\mathcal{L}\{t\}\Big|_{s\to s-3} + \mathcal{L}\{t\}\Big|_{s\to s-4}$

$= \boxed{\frac{1}{(s-2)^2} + \frac{2}{(s-3)^2} + \frac{1}{(s-4)^2}}$ $(s>4)$

$(s>2)\cap(s>3)\cap(s>4)$

⑥ $\mathcal{L}\{t\cos(2t)\} = -\frac{d}{ds}\mathcal{L}\{\cos(2t)\} = -\frac{d}{ds}\frac{s}{s^2+4} =$ $(s>0)$

$$= -\frac{s^2+4-2s^2}{(s^2+4)^2} = \boxed{\frac{s^2-4}{(s^2+4)^2}} \quad (s>0)$$

⑦ $\mathcal{L}\{e^{5t}\sinh(3t)\} = \mathcal{L}\{\sinh(3t)\}\Big|_{s\to s-5} = \frac{3}{s^2-9}\Big|_{s\to s-5}$

$$= \boxed{\frac{3}{(s-5)^2-9}} \quad (s>8)$$

$s>3$

$s-5>3 \quad s>8$

⑧ $\mathcal{L}\{te^{2t}\sin(6t)\} = -\frac{d}{ds}\mathcal{L}\{e^{2t}\sin(6t)\}$

$$= -\frac{d}{ds}\mathcal{L}\{\sin(6t)\}\Big|_{s\to s-2}$$

$$= -\frac{d}{ds}\frac{6}{s^2+36}\Big|_{s\to s-2} = -\frac{d}{ds}\frac{6}{(s-2)^2+36}$$

$(s>0) \mapsto (s-2>0)$

$$= \boxed{\frac{12(s-2)}{((s-2)^2+36)^2}} \quad (s>2)$$

⑨ $\mathcal{L}\{e^{-2t}(t^3+1)^2\} = \mathcal{L}\{(t^3+1)^2\}\Big|_{s\to s+2}$

$$= \mathcal{L}\{t^6+2t^3+1\}\Big|_{s\to s+2}$$

$$= \left(\frac{6!}{s^7} + \frac{2\cdot 3!}{s^4} + \frac{1}{s}\right)\Big|_{s\to s+2}$$

$(s>0) \mapsto (s+2>0)$

$$= \boxed{\frac{6!}{(s+2)^7} + \frac{12}{(s+2)^4} + \frac{1}{s+2}} \quad (s>-2)$$

⑩ $\mathcal{L}\{te^{at}\sin(bt)\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\sin(bt)\} = -\frac{d}{ds}\mathcal{L}\{\sin(bt)\}_{s\to s-a}$

$$= -\frac{d}{ds}\frac{b}{s^2+b^2}\Big|_{s\to s-a} = -\frac{d}{ds}\frac{b}{(s-a)^2+b^2}$$

$s>0$
$\downarrow$
$s-a>0$

$$= \boxed{\frac{2b(s-a)}{((s-a)^2+b^2)^2}} \quad (s>a)$$

(11) $y'' - 2y' + y = 0$; $y(0) = 2$; $y'(0) = 3$. $\qquad \mathcal{L}\{y(t)\} = ?\ \big(Y(s) = ?\big)$

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{0\}$$

$$\underbrace{\mathcal{L}\{y''\}}_{s^2Y(s) - sy(0) - y'(0)} - 2\underbrace{\mathcal{L}\{y'\}}_{sY(s) - y(0)} + \underbrace{\mathcal{L}\{y\}}_{Y(s)} = 0$$

$$s^2 Y(s) - 2s - 3 - 2s\,Y(s) + 4 + Y(s) = 0$$

$$(s^2 - 2s + 1)\,Y(s) = 2s - 1 \Rightarrow \qquad \boxed{Y(s) = \frac{2s-1}{s^2 - 2s + 1}}$$