§. Haar Functions

~ Dyadic Grids/Lattices ~

Def.: The standard dyadic grid on $\mathbb{R}$: $(\mathcal{D}_0)$ the collection of all standard dyadic intervals:

$$\mathcal{D}_0 := \left\{ I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right) : m, k \in \mathbb{Z} \right\}$$

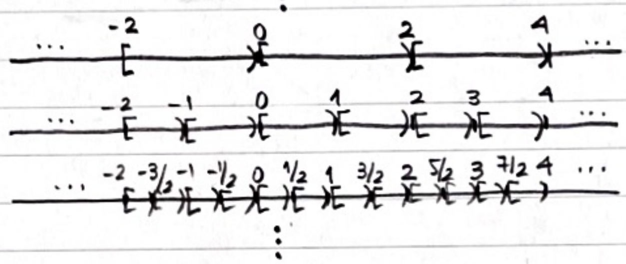
$\vdots$

$k=-1: \{[2m, 2(m+1)) : m \in \mathbb{Z}\}$

$k=0: \{[m, m+1) : m \in \mathbb{Z}\}$

$k=1: \{[\frac{m}{2}, \frac{m+1}{2}) : m \in \mathbb{Z}\}$

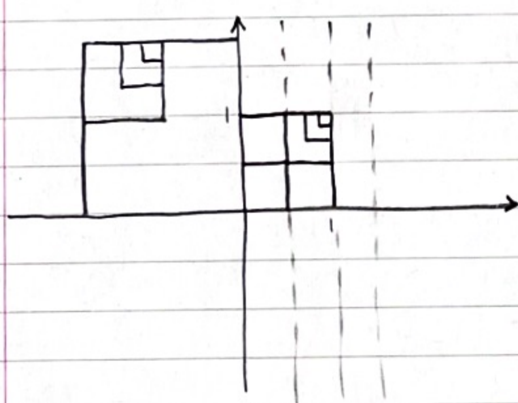
$\vdots$



This easily generalizes to $\mathbb{R}^n$, where instead of dyadic intervals we have dyadic cubes:

$$\mathcal{D}_0 := \left\{ Q = \prod_{j=1}^{n} \left[\frac{m_j}{2^k}, \frac{m_j+1}{2^k}\right) : m_1, \dots, m_n, k \in \mathbb{Z} \right\}$$

These are just products of dyadic intervals of the same length:



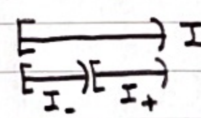
Remark: Each dyadic cube $Q \in \mathcal{D}_0$ has exactly 2^n dyadic children:
For a dyadic cube $Q \in \mathcal{D}_0$ with side length
$$\ell(Q) = 2^{-k} \quad (k \in \mathbb{Z})$$
its dyadic children are the 2^n subcubes $P \in \mathcal{D}_0$ s.t. $P \subsetneq Q$ and $\ell(P) = \frac{\ell(Q)}{2} = 2^{-k-1}$.

In the 1-dim'l case of dyadic intervals, these are given special names:

Def.: Given a dyadic interval $I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)$, its two dyadic children

$$I_- := \left[\frac{m}{2^k} = \frac{2m}{2^{k+1}}, \frac{2m+1}{2^{k+1}}\right) ; \quad I_+ := \left[\frac{2m+1}{2^{k+1}}, \frac{2m+2}{2^{k+1}} = \frac{m+1}{2^k}\right)$$

are known as the left & right halves of I.



Similarly every $Q \in \mathcal{D}_0$ has a unique dyadic parent: for $Q \in \mathcal{D}_0$, $\ell(Q) = 2^{-k}$ $(k \in \mathbb{Z})$, its dyadic parent (sometimes denoted $\hat{Q}$) is the unique dyadic cube $\hat{Q} \in \mathcal{D}_0$ s.t. $\hat{Q} \supset Q$ and $\ell(\hat{Q}) = 2\ell(Q) = 2^{-k+1}$.

The main usefulness of $\mathcal{D}_0$ lies in its geometrical properties, which are used to define a general dyadic grid (aka "dyadic lattice):

Definition: A dyadic grid on $\mathbb{R}^n$ is a collection $\mathcal{D}$ of cubes that satisfy the following 3 properties:

1). Every $Q \in \mathcal{D}$ has side length 2^k, for some $k \in \mathbb{Z}$ (the dyadic part)

$$\ell(Q) = 2^k, \ k \in \mathbb{Z}, \quad \forall Q \in \mathcal{D}$$

2). For a fixed $k_0 \in \mathbb{Z}$, the collection $\{Q \in \mathcal{D} : \ell(Q) = 2^{k_0}\}$ forms a partition of $\mathbb{R}^n$ (i.e. any two dyadic cubes of equal side length are either disjoint or they coincide, and their union is all of $\mathbb{R}^n$).

3). Any two dyadic cubes are either disjoint, or one contains the other:

$$\forall P, Q \in \mathcal{D}, \ (P \cap Q) \in \{\emptyset, P, Q\}.$$

Ex: Every bounded open set in $\mathbb{R}^n$ is the disjoint, countable union of the maximal dyadic cubes contained in it.

Pf: Let $U \subset \mathbb{R}^n$ be a bounded open set.

$x \in U$, U open $\Rightarrow \exists$ ball B around x s.t. $x \in B \subset U \Rightarrow \exists$ dyadic cube $Q \in \mathcal{D}$ s.t. $x \in Q \subset U$.

Let $\mathcal{D}_U^* = \{$maximal dyadic cubes $Q \in \mathcal{D}$ s.t. $Q \subset U\}$ (ok b/c U = bounded)

$\hookrightarrow$ means $\nexists\, Q' \in \mathcal{D}$ s.t. $Q \subsetneq Q' \subset U$

$\Rightarrow U \subset \bigcup_{Q \in \mathcal{D}_U^*} Q$ is obvious from above, and $\bigcup_{Q \in \mathcal{D}_U^*} Q \subset U$ is obvious by definition.

So $U = \bigcup_{Q \in \mathcal{D}_U^*} Q$. The disjoint (useful) part:

Let $P, Q \in \mathcal{D}_U^*$. If $P \cap Q \neq \emptyset$, then either

- $P \subsetneq Q$: contradicts maximality of P in U
- $Q \subsetneq P$: contradicts maximality of Q in U
- $P = Q$: only possibility! ∎