Math 2401

Chapter $\boxed{13}$ Review

- **Vector-valued functions / Curves**

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle x(t), y(t), z(t)\rangle \\ &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\end{aligned}}$$

is called

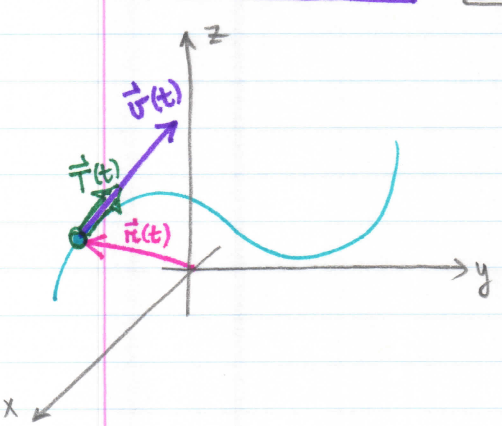
- **Continuous** at t_0 if $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$
- **Differentiable** at t_0 if each of the component functions $x(t), y(t), z(t)$ is differentiable at t_0.
- **Smooth** if the derivative $\vec{r}\,'(t)$ is continuous and never $\vec{0}$.

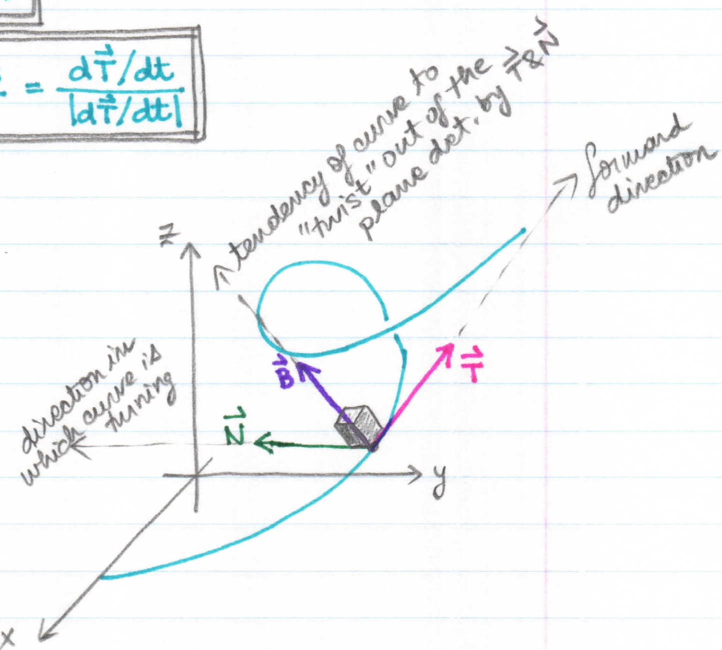
- **Vectors/Quantities associated to $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ (smooth):**

- **Velocity:** $\boxed{\vec{v}(t) = \vec{r}\,'(t) = \frac{d}{dt}\vec{r} = \langle x'(t), y'(t), z'(t)\rangle}$ (derivative of position)
- **Speed:** $\boxed{|\vec{v}(t)|}$
- **Acceleration:** $\boxed{\vec{a}(t) = \frac{d}{dt}\vec{v} = \frac{d^2\vec{r}}{dt^2}}$ (derivative of velocity)
- **Length:** $\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |\vec{v}(t)|\,dt}$ Length of a smooth curve that is traced exactly once as $a \le t \le b$.
- **Arc Length Parameter:** $\boxed{s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\,d\tau}$ Length along the curve, measured from a basepoint $P(t_0) = (x(t_0), y(t_0), z(t_0))$.

$$\boxed{\frac{ds}{dt} = |\vec{v}(t)|} \qquad \boxed{\frac{dt}{ds} = \frac{1}{|\vec{v}(t)|}}$$

- **Unit Tangent Vector:** $\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|}}$; $\boxed{\frac{d\vec{r}}{ds} = \vec{T}}$
- **Curvature:** $\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|}$
- **Unit Normal Vector:** $\boxed{\vec{N} = \frac{d\vec{T}/ds}{\kappa} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}}$
- **Binormal Vector:** $\boxed{\vec{B} = \vec{T} \times \vec{N}}$





- **Tangential & Normal Components of Acceleration:**

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\boxed{\boxed{a_T} = \frac{d^2s}{dt^2} = \boxed{\frac{d}{dt}|\vec{v}(t)|}}$$
— Tangential scalar component of acceleration (measures how much of $\vec{a}$ is acting in the direction of motion)

$$\boxed{\boxed{a_N} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}(t)|^2 = \boxed{\sqrt{|\vec{a}|^2 - a_T^2}}}$$
→ Normal scalar component of $\vec{a}$ (measures how much of $\vec{a}$ is acting normal to the motion)

- **Torsion:** $\boxed{\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}}$

- **Differentiation Rules for Vector Functions:**

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \left(\frac{d}{dt}\vec{u}\right) \cdot \vec{v} + \vec{u} \cdot \left(\frac{d}{dt}\vec{v}\right)$$

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \left(\frac{d}{dt}\vec{u}\right) \times \vec{v} + \vec{u} \times \left(\frac{d}{dt}\vec{v}\right)$$

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\, \vec{u}'(f(t))$$

- **Vector Functions of Constant Length:**

If $\vec{r}(t)$ is a differentiable function of t with constant length, then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal at all t:

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0.$$