

14.1 Functions of Several Variables

- Domain & Range
- Interior Point, Boundary Point
- Open Set, Closed Set, Bounded Set
- Level Curves: $f(x,y) = c$
- Level Surfaces: $f(x,y,z) = c$

14.2 Limits & Continuity in Higher Dimensions

- Major difference from Calculus I limits (one variable): a point (x,y) can approach a point (x_0, y_0) in the plane from infinitely many directions, along infinitely many paths. They must all agree in order for the limit to exist!
- Two-Path Test: If $f(x,y)$ has 2 different limits along 2 different paths in the domain as (x,y) approaches (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y)$ does not exist.

• Typical Examples:

- ① Continuous functions (aka "plug it in, nothing bad happens"):

$$\lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \boxed{-3}$$

- ② % limits where we factor & cancel terms

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x(x-y)}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} - \sqrt{y}} = \boxed{0}$$

- ③ Two-Path Test (Limit DNE):

$$f(x,y) = \frac{2x^2y}{x^4 + y^2} \text{ has no limit as } (x,y) \rightarrow (0,0)$$

(approach along lines) $f(x,y)|_{y=kx} = \frac{2x^2(kx)}{x^4 + (kx)^2} = \frac{2kx^3}{x^4 + k^2x^2} = \frac{2kx}{x^2 + k^2} \xrightarrow{x \rightarrow 0} \boxed{0}$ ↑ inconclusive

(approach along parabolas) $f(x,y)|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2} \xrightarrow{x \rightarrow 0} \boxed{\frac{2k}{1+k^2}}$

Taking $k=0$ and $k=2$, for example, $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ DNE by the 2-Path Test.

④ Switching to Polar Coord. (to show that a limit does exist).

$$\lim_{(x,y) \rightarrow (0,0)} \left(xy \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim_{r \rightarrow 0} \left((r \cos \theta)(r \sin \theta) \frac{(r \cos \theta)^2 - (r \sin \theta)^2}{(r \cos \theta)^2 + (r \sin \theta)^2} \right)$$

$$= \lim_{r \rightarrow 0} r^2 \underbrace{\cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta)}_{\text{bounded}}$$

$$= \boxed{0}$$

$$\lim_{(x,y) \rightarrow (0,0)} \left[\ln \left(\frac{7x^2 - x^2y^2 + 7y^2}{x^2 + y^2} \right) \right] = \lim_{r \rightarrow 0} \left[\ln \frac{7r^2 \cos^2 \theta - r^4 \cos^2 \theta \sin^2 \theta + 7r^2 \sin^2 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \right]$$

$$= \lim_{r \rightarrow 0} \left[\ln \frac{7r^2 (\cos^2 \theta + \sin^2 \theta) - r^4 \cos^2 \theta \sin^2 \theta}{r^2} \right]$$

$$= \lim_{r \rightarrow 0} \left[\ln \left(7 - \underbrace{r^2 \cos^2 \theta \sin^2 \theta}_{\text{bounded}} \right) \right] = \boxed{\ln(7)}$$

$\downarrow r \rightarrow 0$
0

14.3 Partial Derivatives

Partial derivative $\frac{\partial f}{\partial x}(x_0, y_0)$:

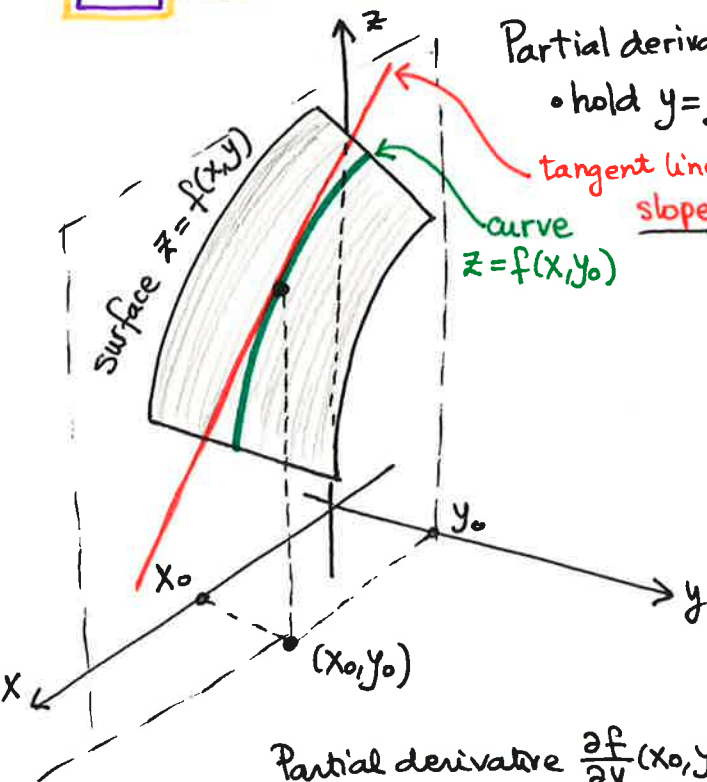
• hold $y = y_0$ constant

tangent line to this curve

slope:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

curve $z = f(x, y_0)$



Partial derivative $\frac{\partial f}{\partial y}(x_0, y_0)$:

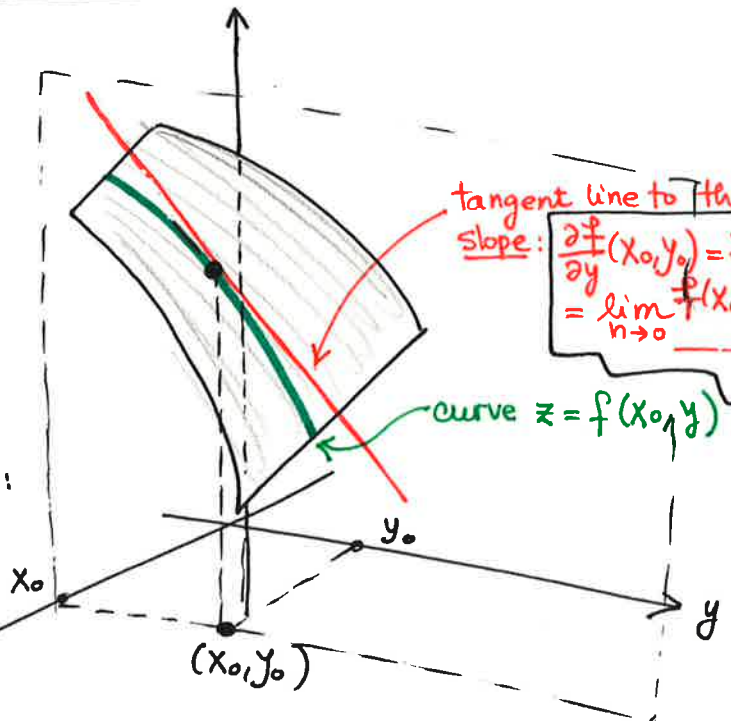
• hold $x = x_0$ constant

tangent line to this curve

slope:

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

curve $z = f(x_0, y)$



• To compute $\frac{\partial f}{\partial x}$: hold the other variables (y, z etc.) constant & differentiate by x .

Ex): $f(x, y, z) = x^2 y + x \sin(yz) + \cos(xyz)$

To find $\frac{\partial f}{\partial x}$, think of f as: $f(x, y, z) = x^2 y + x \sin(yz) + \cos(xyz)$
(green = variable)

$$\frac{\partial f}{\partial x} = 2xy + \sin(yz) + (-\sin(xyz) \cdot yz)$$

To find $\frac{\partial f}{\partial y}$, think of f as: $f(x, y, z) = x^2 y + x \sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial y} = x^2 + x \cos(yz) \cdot z + (-\sin(xyz) \cdot xz)$$

To find $\frac{\partial f}{\partial z}$, think of f as: $f(x, y, z) = x^2 y + x \sin(yz) + \cos(xyz)$

$$\frac{\partial f}{\partial z} = x \cos(yz) \cdot y + (-\sin(xyz) \cdot xy)$$

• Higher Order Partial Derivatives:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad ; \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \quad ; \quad \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad ;$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = f_{zyx} \quad ; \quad \frac{\partial^4 f}{\partial x \partial y^3} = f_{yyyx} \quad ; \quad \text{etc. etc.}$$

• Mixed Derivative Theorem:

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0) , then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

14.4 The Chain Rule:

$$w = w(x, y, z)$$

$$x = x(t)$$

$$y = y(t)$$

$$z = z(t)$$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= w_x \cdot x'(t) + w_y \cdot y'(t) + w_z \cdot z'(t) \end{aligned}$$

$$w = w(x, y, z)$$

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\ &= w_x \cdot x_u + w_y \cdot y_u + w_z \cdot z_u \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\ &= w_x \cdot x_v + w_y \cdot y_v + w_z \cdot z_v \end{aligned}$$

• Implicit Differentiation Simplified: If $F(x,y)=0$ defines y implicitly as a differentiable function of x , then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Ex.: $6x^3 - 7y^2 - xy = 0$; find $\frac{dy}{dx}$

Old way: $\frac{d}{dx}(6x^3 - 7y^2 - xy) = \frac{d}{dx}(0)$
 $18x^2 - 14y\left(\frac{dy}{dx}\right) - y - x\left(\frac{dy}{dx}\right) = 0$
 $18x^2 - y = (x + 14y)\frac{dy}{dx}$
 $\frac{18x^2 - y}{x + 14y} = \frac{dy}{dx}$

New way: $F(x,y) = 6x^3 - 7y^2 - xy$
 $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{18x^2 - y}{-14y - x} = \frac{18x^2 - y}{14y + x}$

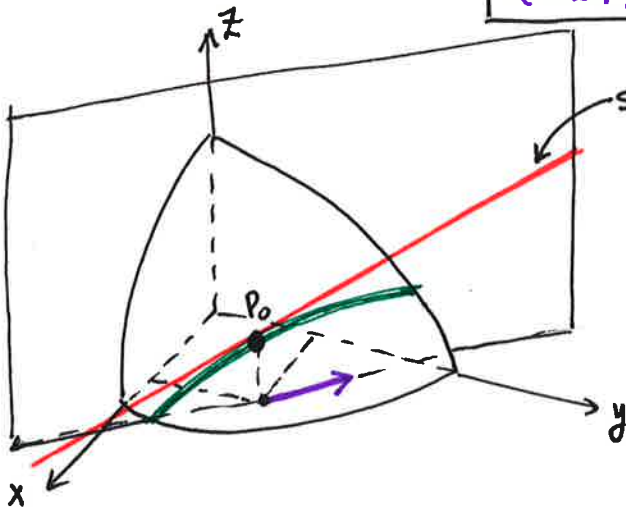
14.5 Directional Derivatives & Gradient Vectors

$f(x,y,z)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$f(x,y)$; Gradient Vector: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Directional Derivative of f at P_0 in the direction of the unit vector \vec{u} :

$$(D_{\vec{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$



slope = $(D_{\vec{u}}f)_{P_0}$ = rate of change in the direction of \vec{u} .

$$D_{\vec{u}}f = |\nabla f| |\vec{u}| \cos\theta = |\nabla f| \cos\theta$$

- f \uparrow most rapidly in the direction of ∇f ($\theta = 0$); directional deriv. there is $|\nabla f|$.
- f \downarrow most rapidly in the direction of $-\nabla f$ ($\theta = \pi$); directional deriv. there is $-|\nabla f|$.

14.6 Tangent Planes & Normal Lines

Tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$:

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$$

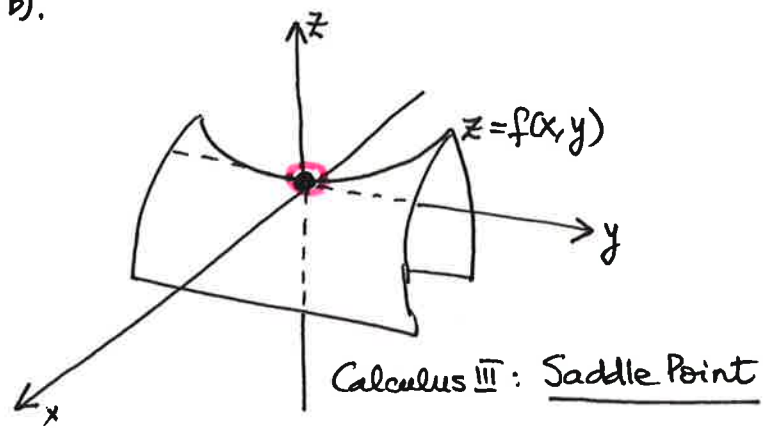
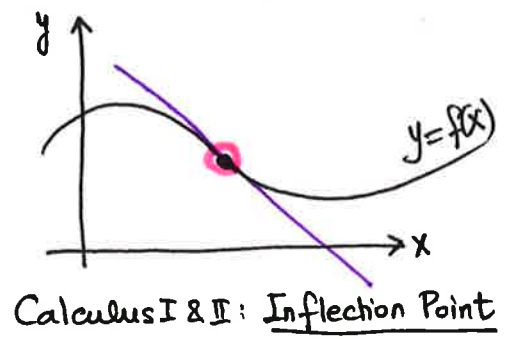
Normal line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$:

$$\begin{aligned} x &= x_0 + f_x(P_0)t \\ y &= y_0 + f_y(P_0)t \\ z &= z_0 + f_z(P_0)t \end{aligned}$$

14.7 Extreme Values & Saddle Points

Def.: A point (a,b) is called a critical point of $f(x,y)$ if (a,b) is an interior point to the domain of f and either $f_x(a,b) = f_y(a,b) = 0$ or one or both f_x, f_y do not exist at (a,b) .

Def.: A saddle point for $f(x,y)$ is a critical point (a,b) such that in every open disk centered at (a,b) there are domain points (x,y) such that $f(x,y) > f(a,b)$ and points where $f(x,y) < f(a,b)$.



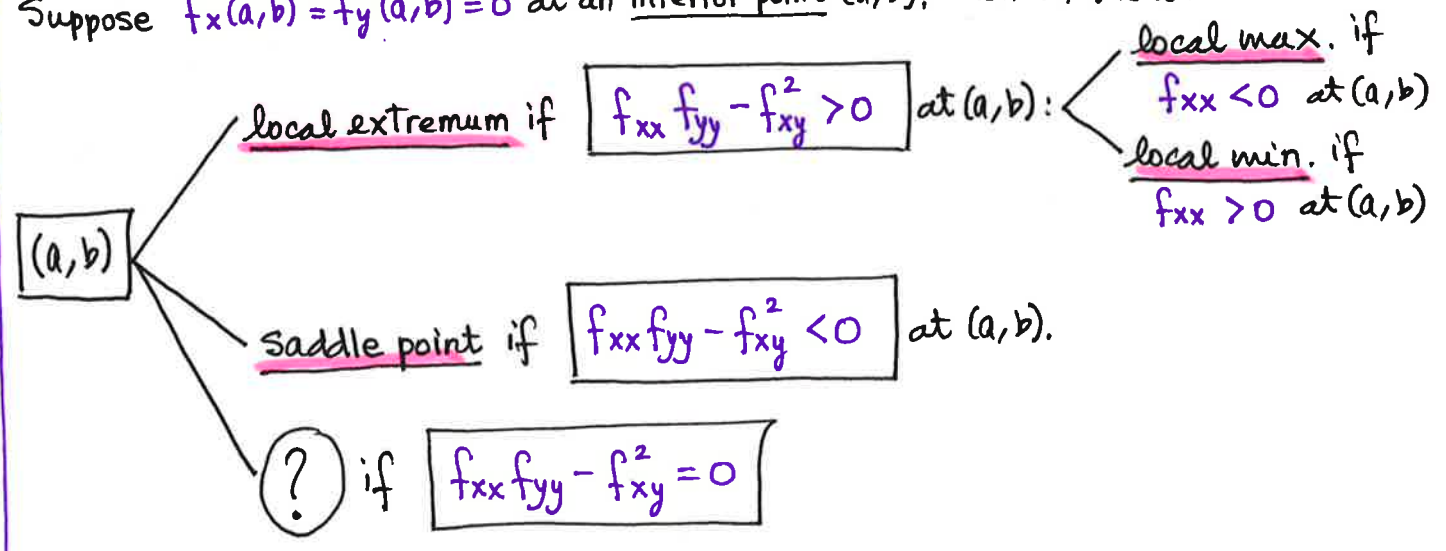
First Derivative Test:

If $f(x,y)$ has a local min. or max at an interior point (a,b) and $f_x(a,b), f_y(a,b)$ exist, then $f_x(a,b) = f_y(a,b) = 0$.

In other words: local extrema occur at critical points or at boundary points.

Second Derivative Test:

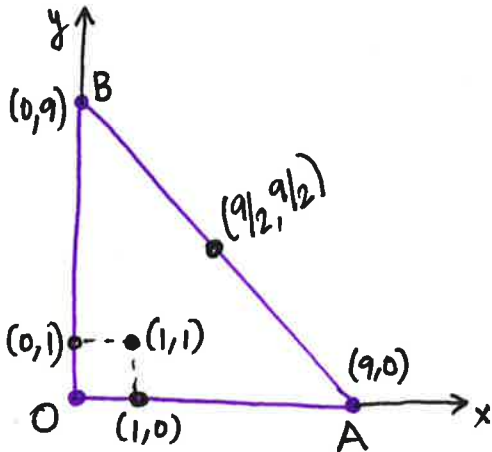
Suppose $f_x(a,b) = f_y(a,b) = 0$ at an interior point (a,b) . Then (a,b) is a:



Min & Max on Closed Bounded Regions :

- ① Find the critical points interior to the region & evaluate f there.
- ② Find the boundary points where f has local extrema & evaluate f there.
- ③ Look through the lists & find the absolute min & max.

Ex]: $f(x,y) = 2+2x+2y-x^2-y^2$; Triangular region in Quad I bounded by $x=0, y=0, y=9-x$.



List :

$$f(1,1) = 4 \rightarrow \text{Max}$$

$$f(1,0) = 3$$

$$f(0,0) = 2$$

$$f(9,0) = -61 \rightarrow \text{Min}$$

$$f(0,1) = 3$$

$$f(0,9) = -61$$

$$f(9/2, 9/2) = -41/2$$

① Interior points : $f_x = 2-2x$; $f_y = 2-2y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \begin{cases} 2-2x = 0 \\ 2-2y = 0 \end{cases} \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Critical point: $(1,1)$

Is it inside the region? Yes. So evaluate f & list it.

② Boundary Points :

i) OA : $y=0$

$$f(x,0) = 2+2x-x^2, \quad 0 \leq x \leq 9$$

Extreme values occur @ boundary pts. or critical points in $(0,9)$

$$f'(x,0) = 2-2x$$

Critical point : $(1,0)$ - evaluate f here and at the boundary points $(0,0), (9,0)$.

ii) OB : $x=0$

$$f(0,y) = 2+2y-y^2, \quad 0 \leq y \leq 9$$

$$f'(0,y) = 2-2y$$

Critical point : $(0,1)$; Boundary points: $(0,0), (0,9)$

↑
already done

iii) AB : $y=9-x$

Already evaluated endpoints $(0,9)$ & $(9,0)$, so we only need to look for critical points.

$$\begin{aligned} f(x,9-x) &= 2+2x+2(9-x)-x^2-(9-x)^2 \\ &= -61+18x-2x^2 \end{aligned}$$

$$f'(x,9-x) = 18-4x$$

$$18-4x=0 \Rightarrow x=9/2 \Rightarrow y=9-9/2=9/2$$

Critical point : $(9/2, 9/2)$

14.8 Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq \vec{0}$ when $g(x,y,z)=0$.
To find local min & max values of f subject to the constraint $g(x,y,z)=0$

Find the values of x, y, z and λ that satisfy:

$$\begin{cases} \nabla f = \lambda(\nabla g) \\ g(x,y,z) = 0 \end{cases}$$

Two constraints: To find local extrema of a differentiable $f(x,y,z)$ subject to the constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$ and g_1, g_2 are differentiable (∇g_1 not parallel to ∇g_2), find the values of $x, y, z, \lambda_1, \lambda_2$ that satisfy:

$$\begin{cases} \nabla f = \lambda_1(\nabla g_1) + \lambda_2(\nabla g_2) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$$

Example: Find the max. value of $f(x,y) = 58 - x^2 - y^2$ on the line $x + 7y = 50$.

$$g(x,y) = x + 7y - 50$$

$$\nabla f = \langle -2x, -2y \rangle$$

$$\nabla g = \langle 1, 7 \rangle$$

$$\text{Solve: } \begin{cases} -2x = \lambda \\ -2y = 7\lambda \\ x + 7y - 50 = 0 \end{cases} \quad \begin{cases} x = -\lambda/2 \\ y = -7\lambda/2 \\ x + 7y - 50 = 0 \end{cases}$$

$$-\frac{\lambda}{2} + 7\left(-\frac{7\lambda}{2}\right) - 50 = 0$$

$$\lambda + 49\lambda + 100 = 0$$

$$50\lambda = -100$$

$$\lambda = -2$$

$$\Rightarrow x = +1, y = +7$$

The extreme value occurs at $(1, 7)$, where:

$$f(1, 7) = \boxed{8}.$$