Chapter 16 - Review

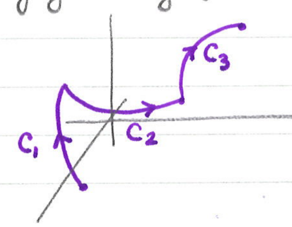
16.1 Line Integrals of Scalar Functions

- If $C: [\vec{r}(t), a \le t \le b]$ is a smooth curve and f is continuous on C, the line (path) integral of f along C is:

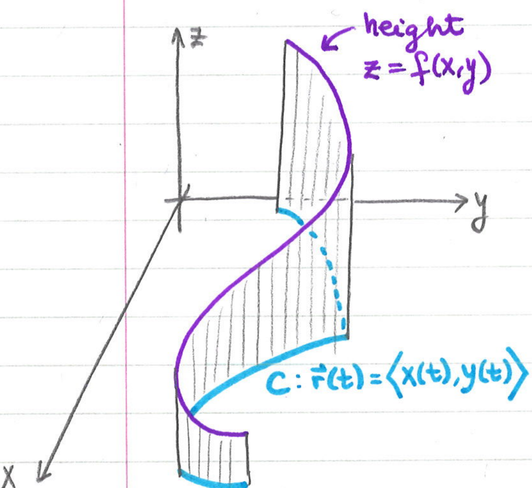
$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{v}(t)|\,dt}$$

- Additivity: If C is a piecewise smooth curve (made by joining finitely many smooth curves $C_1, \ldots, C_n$ end to end), then:

$$\int_C f\,ds = \int_{C_1} f\,ds + \ldots + \int_{C_n} f\,ds$$



- Geometrical Interpretation of line integrals in the plane:



- $C: [\vec{r}(t), a \le t \le b]$ smooth curve in the plane $\vec{r}(t) = \langle x(t), y(t) \rangle$
- $f(x,y)$ – non-negative continuous function
- Area of "winding wall" – part of the cylindrical surface $(x(t), y(t), z)$ bounded by C and $z = f(x,y)$:

$$\boxed{\int_C f\,ds}$$

Example: $f(x,y,z) = x + \sqrt{y} - z^4$; $C: \vec{r}(t) = \langle t, t^2, 0 \rangle$, $0 \le t \le 2$.

$x = t,\ y = t^2,\ z = 0$

$f(\vec{r}(t)) = t + \sqrt{t^2} - 0^4 = 2t$ (because $t \ge 0$)

$\vec{v}(t) = \langle 1, 2t, 0 \rangle$

$|\vec{v}(t)| = \sqrt{1 + 4t^2}$

$$\int_C f\,ds = \int_0^2 f(\vec{r}(t))\,|\vec{v}(t)|$$

$$= \int_0^2 2t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{4} \cdot \frac{2}{3}(1+4t^2)^{3/2}\Big|_0^2$$

$$= \boxed{\frac{1}{6}\left(17^{3/2} - 1\right)}$$

16.2 Line Integrals of Vector Fields

- A vector field is a function that assigns a vector to each point in its domain.
  - (2D): $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$
  - (3D): $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$
- A vector field is continuous / differentiable if each component function is.
- Examples:
  - Velocity field (fluid flowing through a region)
  - Gravitational field (or more generally, force fields)
  - Gradient fields: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

- Line Integral of a Continuous Vector Field $\vec{F} = \langle M, N, P \rangle$ along a smooth curve C: $\vec{r}(t)$, $a \le t \le b$:

$$\boxed{\begin{aligned} \int_C \vec{F}\cdot\vec{T}\,ds &= \int_C \vec{F}\cdot d\vec{r} \\ &= \int_a^b \left(\vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\right) dt \\ &= \int_C M\,dx + N\,dy + P\,dz \end{aligned}}$$

- Special Interpretations of $\int_C \vec{F}\cdot d\vec{r}$:
  - If $\vec{F}$ is a force field, $\int_C \vec{F}\cdot d\vec{r}$ is the work done by the force field to move an object along C.
  - If $\vec{F}$ is a velocity field of a fluid flowing through space (or plane) $\int_C \vec{F}\cdot d\vec{r}$ is called the flow of $\vec{F}$ along C.
    - If C is a closed curve, $\int_C \vec{F}\cdot d\vec{r}$ is called the circulation along the curve, and is denoted: $\boxed{\oint_C \vec{F}\cdot d\vec{r}}$ or $\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds}$.

Example: Find the work done by $\vec{F} = \langle 2xy, 4y, -yz \rangle$ over $C: \vec{r}(t) = \langle t, t^2, t \rangle$, $0 \le t \le 1$.

$x = t$; $dx = dt$; $M = 2xy = 2t \cdot t^2 = 2t^3$
$y = t^2$; $dy = 2t\,dt$; $N = 4y = 4t^2$
$z = t$; $dz = dt$; $P = -yz = -t^2 \cdot t = -t^3$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$
$$= \int_0^1 \left(2t^3 + 4t^2 \cdot 2t - t^3\right) dt = \int_0^1 9t^3\,dt = \frac{9t^4}{4}\Big|_0^1 = \boxed{9/4}.$$

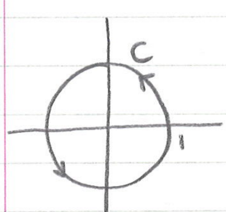
Flux across a Simple Closed Curve

- Let $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ be a continuous vector field in the plane and C is a smooth, simple, positively oriented, closed curve in the domain of $\vec{F}$.

The flux of $\vec{F}$ along C: $\boxed{\oint_C \vec{F} \cdot \vec{n}\,ds = \oint_C M\,dy - N\,dx}$

(where $\vec{n} = \vec{T} \times \vec{k}$ is the outward normal vector on C).

Example: Find the flux of $\vec{F} = \langle x-y, x \rangle$ over $C: x^2 + y^2 = 1$.



$C: \vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$x = \cos t$; $dx = -\sin t\,dt$; $M = x - y = \cos t - \sin t$
$y = \sin t$; $dy = \cos t\,dt$; $N = x = \cos t$

$$\text{Flux} = \oint_C M\,dy - N\,dx = \int_0^{2\pi} \left[(\cos t - \sin t)(\cos t) - (\cos t)(-\sin t)\right] dt$$
$$= \int_0^{2\pi} \left(\cos^2 t - \cancel{\sin t \cos t} + \cancel{\sin t \cos t}\right) dt$$
$$= \int_0^{2\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt = \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{\pi}.$$

16.3 Conservative Fields

- A vector field $\vec{F}$ is said to be conservative on a domain D if for any two points A and B in D, the line integral $\int_C \vec{F}\cdot d\vec{r}$ is the same along all paths C in D from A to B.
- In this case, we say the integral $\int_C \vec{F}\cdot d\vec{r}$ is called path independent, and is denoted:

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$$

- We say f is a potential function for a vector field $\vec{F}$ if $\boxed{\vec{F} = \nabla f}$.

- **Fundamental Theorem for Line Integrals:**

  If C is a smooth curve joining A and B in a domain D, parametrized by $\vec{r}(t)$, and $\vec{F} = \nabla f$ for a differentiable function f on D:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)}$$

- **Theorem:** $\boxed{\vec{F} \text{ is a conservative field}} \iff \boxed{\vec{F} = \nabla f}$ for some diff'ble function f.

- **Loop Property:** $\boxed{\vec{F} \text{ is a conservative field}} \iff \boxed{\oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for any loop } C}$

- **Component Test for Conservative Fields:**

$$\boxed{\text{A vector field } \vec{F} = \langle M, N, P\rangle \text{ is conservative}} \iff \boxed{\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

- A differential form $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$ is said to be exact if:

$$\boxed{M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$$

  for some scalar function f.

- **Component Test for Exactness:**

$$\boxed{M\,dx + N\,dy + P\,dz \text{ is exact}} \iff \boxed{\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

Example: $\vec{F} = \langle 2x, 7y, 5z \rangle$

a). Find a potential function for $\vec{F}$ if one exists:

$$\frac{\partial f}{\partial x} = 2x \Rightarrow f = x^2 + g(y,z)$$

$$\left.\begin{array}{l} \Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial y} \\ \dfrac{\partial f}{\partial y} = 7y \end{array}\right\} \Rightarrow g = 7y^2/2 + h(z) \Rightarrow f = x^2 + \frac{7y^2}{2} + h(z)$$

$$\left.\begin{array}{l} \Rightarrow \dfrac{\partial f}{\partial z} = h'(z) \\ \dfrac{\partial f}{\partial z} = 5z \end{array}\right\} \Rightarrow h(z) = \frac{5z^2}{2} + C$$

$$\boxed{f(x,y,z) = x^2 + \frac{7y^2}{2} + \frac{5z^2}{2} + C}$$

b). Find $\int_C \vec{F} \cdot d\vec{r}$, where C is a path from $(0,0,0)$ to $(1,1,1)$.

By the Fundamental Theorem: $\int_C \vec{F} \cdot d\vec{r} = f(1,1,1) - f(0,0,0) = 1 + \frac{7}{2} + \frac{5}{2} = \boxed{7}$.

Example: Find a potential function for the vector field:

$$\vec{F} = \left\langle \ln x + \sec^2(8x+8y),\ \sec^2(8x+8y) + \frac{11y}{y^2+z^2},\ \frac{11z}{y^2+z^2} \right\rangle.$$

$$\frac{\partial f}{\partial x} = \ln x + \sec^2(8x+8y) \Rightarrow f = x\ln x - x + \frac{1}{8}\tan(8x+8y) + g(y,z)$$

$$\left.\begin{array}{l} \Rightarrow \dfrac{\partial f}{\partial y} = \sec^2(8x+8y) + \dfrac{\partial g}{\partial y} \\ \dfrac{\partial f}{\partial y} = \sec^2(8x+8y) + \dfrac{11y}{y^2+z^2} \end{array}\right\} \Rightarrow g = \frac{11}{2}\ln(y^2+z^2) + h(z)$$

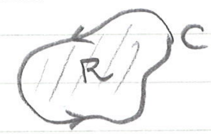
$$\Rightarrow f = x\ln x - x + \frac{1}{8}\tan(8x+8y) + \frac{11}{2}\ln(y^2+z^2) + h(z)$$

$$\left.\begin{array}{l} \Rightarrow \dfrac{\partial f}{\partial z} = \dfrac{11z}{y^2+z^2} + h'(z) \\ \dfrac{\partial f}{\partial z} = \dfrac{11z}{y^2+z^2} \end{array}\right\} \Rightarrow h(z) = C$$

$$\boxed{f = x\ln x - x + \frac{1}{8}\tan(8x+8y) + \frac{11}{2}\ln(y^2+z^2) + C}$$

16.4 Green's Theorem in the Plane

Let C be a piecewise smooth, positively oriented, simple closed curve enclosing a region R in the plane.



Let $\vec{F} = M\vec{i} + N\vec{j}$ be a vector field such that M and N have continuous first partial derivatives in an open region containing R. Then:

(outward) flux of $\vec{F}$ along C: $\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA}$

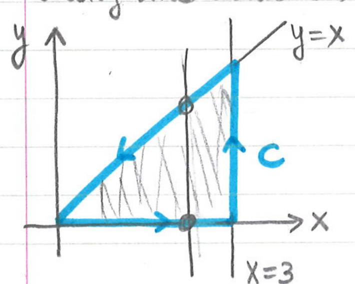
(counterclockwise) circulation of $\vec{F}$ along C: $\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA}$

Computing Area using Green's Theorem:

Suppose a region R in the plane is enclosed by a simple closed curve C that is piecewise smooth. Then:

$$\boxed{\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Example: Find the flux and circulation of $\vec{F} = \langle 7y^2 - 4x^2, 4x^2 + 7y^2 \rangle$ along the curve C in the picture.



Flux: $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$M = 7y^2 - 4x^2, \quad N = 4x^2 + 7y^2$

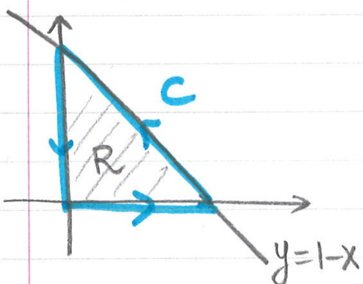
$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^3\int_0^x (-8x + 14y)\,dy\,dx$$

$$= \int_0^3 \left(-8xy + 7y^2\right)\Big|_{y=0}^{y=x} dx = \int_0^3 (-x^2)\,dx = \frac{-x^3}{3}\Big|_0^3 = \boxed{-9}$$

Circulation: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$= \int_0^3\int_0^x (8x - 14y)\,dy\,dx = \boxed{9}.$$

Example: Find $\oint_C 2y^2dx + 2x^2dy$, where C is the triangle bounded by $x=0$, $x+y=1$, $y=0$.



$$M = 2y^2 \; ; \; N = 2x^2$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \qquad \text{(Green Thm.)}$$

$$= \int_0^1 \int_0^{1-x} (4x - 4y)\, dy\, dx$$

$$= \int_0^1 \left(4xy - 2y^2\right)\Big|_{y=0}^{y=1-x} dx = \int_0^1 \left(8x - 2 - 6x^2\right) dx = \left(4x^2 - 2x - 2x^3\right)\Big|_0^1 = \boxed{0}.$$

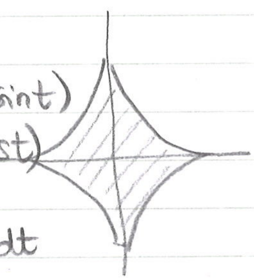
Example: Find the area of the region enclosed by an astroid:

$$C: \vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \; 0 \le t \le 2\pi$$

Using Green's formula:

$$x = \cos^3 t \; ; \; dx = 3\cos^2 t(-\sin t)$$

$$y = \sin^3 t \; ; \; dy = 3\sin^2 t\,(\cos t)$$



$$\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \left((\cos^3 t)(3\sin^2 t \cos t) + (\sin^3 t)(3\cos^2 t \sin t)\right) dt$$

$$= \frac{1}{2}\int_0^{2\pi} 3\sin^2 t \cos^2 t \left(\cos^2 t + \sin^2 t\right) dt$$

$$= \frac{3}{2}\,\frac{1}{4}\int_0^{2\pi} \left(2\sin t \cos t\right)^2 dt$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2(2t)\, dt$$

$$= \frac{3}{16}\int_0^{2\pi} \left(1 - \cos(4t)\right) dt$$

$$= \frac{3}{16}\left(t - \frac{1}{4}\sin(4t)\right)\Big|_0^{2\pi} = \boxed{\frac{3\pi}{8}}$$

16.5 Surfaces and Area

- Parametrization of surfaces:

$$\boxed{\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}} \quad a \le u \le b,\ c \le v \le d$$

- Partial derivatives of $\vec{r}$ with respect to u, v:

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right\rangle \qquad \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right\rangle$$

- A parametrized surface as above is smooth if both $\vec{r}_u$ and $\vec{r}_v$ are continuous, and $\vec{r}_u \times \vec{r}_v$ is never zero.

Area of smooth surface $\vec{r}(u,v)$, $a \le u \le b$, $c \le v \le d$:

$$\boxed{A = \int_c^d \int_a^b \left| \vec{r}_u \times \vec{r}_v \right| \, du\, dv}$$

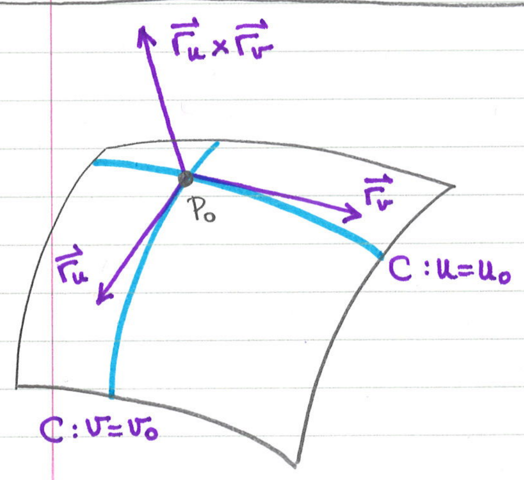
- Surface area differential: $\boxed{d\sigma = \left|\vec{r}_u \times \vec{r}_v\right| du\,dv}$

- Implicitly defined surface: $f(x,y,z) = c$ (level surface)

Area of the surface $f(x,y,z)=c$ over a closed and bounded plane region R:

$$\boxed{A = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} \, dA}$$

where $\vec{p}$ is one of $\vec{i}, \vec{j}, \vec{k}$ (normal to R) and $\nabla F \cdot \vec{p} \neq 0$.

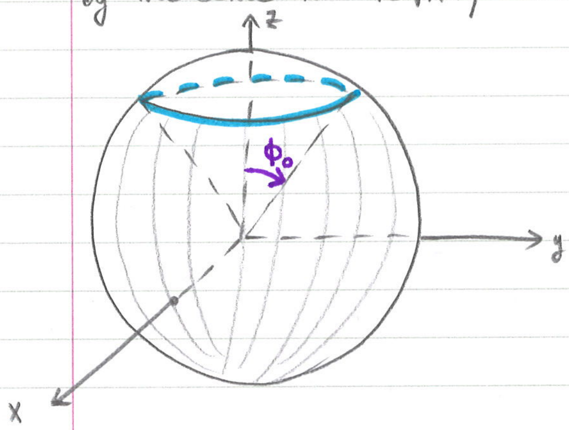


- At every point $P_0 = \vec{r}(u_0, v_0)$ on a parametrized surface $\vec{r}(u,v)$, the vector

$$(\vec{r}_u \times \vec{r}_v)_{P_0}$$

is normal to the surface, and so is normal to the plane tangent to the surface at P_0.

Example: Find the area of the lower portion cut from the sphere $x^2+y^2+z^2=16$ by the cone $z=\sqrt{3}\sqrt{x^2+y^2}$



- Parametrization of sphere:

$$\vec{r}(\phi,\theta)=\langle 4\sin\phi\cos\theta,\ 4\sin\phi\sin\theta,\ 4\cos\phi\rangle$$
$$0\leq\theta\leq 2\pi\ ;\ 0\leq\phi\leq\pi$$

- Intersection with cone (to find ϕ_0):

$z=\sqrt{3}\sqrt{x^2+y^2}$ becomes:

$$4\cos\phi=\sqrt{3}\sqrt{16\sin^2\phi\cos^2\theta+16\sin^2\phi\sin^2\theta}$$
$$=\sqrt{3}\cdot 4\sin\phi \quad\Rightarrow\quad \cos\phi=\sqrt{3}\sin\phi \quad\Rightarrow \boxed{\phi=\pi/6}$$

- Parametrization of lower portion:

$$\boxed{\vec{r}(\phi,\theta)=\langle 4\sin\phi\cos\theta,\ 4\sin\phi\sin\theta,\ 4\cos\phi\rangle\ ,\ 0\leq\theta\leq 2\pi,\ \pi/6\leq\phi\leq\pi}$$

- Surface differential:

$$\vec{r}_\phi=\langle 4\cos\phi\cos\theta,\ 4\cos\phi\sin\theta,\ -4\sin\phi\rangle$$
$$\vec{r}_\theta=\langle -4\sin\phi\sin\theta,\ 4\sin\phi\cos\theta,\ 0\rangle$$
$$\vec{r}_\phi\times\vec{r}_\theta=\langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\rangle$$
$$|\vec{r}_\phi\times\vec{r}_\theta|=16\sqrt{\sin^4\phi+\sin^2\phi\cos^2\phi}=16\sin\phi$$

- Surface Area:

$$\int_0^{2\pi}\int_{\pi/6}^{\pi}16\sin\phi\,d\phi\,d\theta=2\pi\left(-16\cos\phi\right)\Big|_{\pi/6}^{\pi}=2\pi\left(16+16\cdot\frac{\sqrt{3}}{2}\right)$$
$$=\boxed{2\pi\left(16+8\sqrt{3}\right)}$$

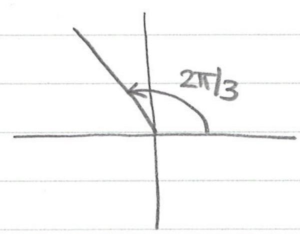
Example: Equation for tangent plane to circular cylinder:

$$\vec{r}(\theta, z) = \left\langle 4\sin(2\theta),\ 8\sin^2\theta,\ z \right\rangle$$

at $P_0(2\sqrt{3}, 6, 1)$, corresponding to $(\theta, z) = (\pi/3, 1)$.

$$\vec{r}_\theta = \left\langle 8\cos(2\theta),\ 16\sin\theta\cos\theta,\ 0 \right\rangle = \left\langle 8\cos(2\theta),\ 8\sin(2\theta),\ 0 \right\rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$



$$\vec{r}_\theta \times \vec{r}_z = \left\langle 8\sin(2\theta),\ -8\cos(2\theta),\ 0 \right\rangle$$

Vector normal to the tangent plane at P_0:

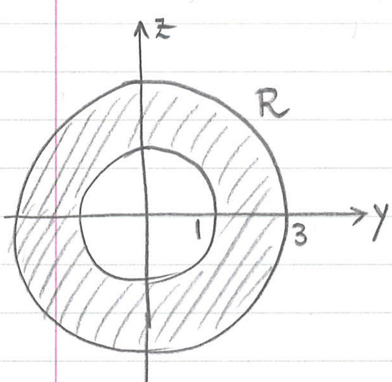
$$\vec{n} = \left(\vec{r}_\theta \times \vec{r}_z\right)\Big|_{(\pi/3, 1)} = \left\langle 8\sin\left(\tfrac{2\pi}{3}\right),\ -8\cos\left(\tfrac{2\pi}{3}\right),\ 0 \right\rangle$$

$$= \left\langle 8 \cdot \tfrac{\sqrt{3}}{2},\ -8 \cdot \left(-\tfrac{1}{2}\right),\ 0 \right\rangle = \left\langle 4\sqrt{3}, 4, 0 \right\rangle.$$

Equation of plane: $4\sqrt{3}(x - 2\sqrt{3}) + 4(y-6) + 0(z-1) = 0$

$$\boxed{4\sqrt{3}x - 4y = 48}$$

Example: Find the area of the portion of the paraboloid $x = 10 - y^2 - z^2$ that lies above the ring $1 \le y^2 + z^2 \le 9$ in the yz-plane.



$f = x + y^2 + z^2$; $\nabla f = \langle 1, 2y, 2z \rangle$

$|\nabla f| = \sqrt{1 + 4y^2 + 4z^2}$; $\vec{p} = \vec{i}$

$\nabla f \cdot \vec{i} = 1$

$$\text{Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{i}|}\, dA = \iint_R \frac{\sqrt{1 + 4y^2 + 4z^2}}{1}\, dA$$

$$= \int_0^{2\pi} \int_1^3 \sqrt{1 + 4r^2} \cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \left(\frac{1}{8} \cdot \frac{2}{3}(1 + 4r^2)^{3/2}\right)\Bigg|_1^3 = \frac{\pi}{6}\left(49^{3/2} - 1\right) = \boxed{57\pi}$$

16.6 Surface Integrals

	Parametric Surface	Implicit Surface
	$S: \vec{r}(u,v)$; (u,v) in R	$S: f(x,y,z) = c$ (level surface) lying over closed & bounded "shadow" region R in one of the coordinate planes.
Surface area differential	$d\sigma = \lvert \vec{r}_u \times \vec{r}_v \rvert \, du\, dv$	$d\sigma = \dfrac{\lvert\nabla f\rvert}{\lvert \nabla f \cdot \vec{p} \rvert} dA$ (where $\vec{p}$ is one of $\vec{i}, \vec{j}, \vec{k}$, normal to R, and dA is the area differential on R).
Integrate a function $G(x,y,z)$ over surface S	$\iint_S G\, d\sigma = \iint_R G(\vec{r}(u,v)) \lvert \vec{r}_u \times \vec{r}_v \rvert \, du\, dv$	$\iint_S G\, d\sigma = \iint_R G(x,y,z) \dfrac{\lvert\nabla f\rvert}{\lvert \nabla f \cdot \vec{p}\rvert} dA$
Unit normal vector field	$\vec{n} = \pm \dfrac{\vec{r}_u \times \vec{r}_v}{\lvert \vec{r}_u \times \vec{r}_v \rvert}$	$\vec{n} = \pm \dfrac{\nabla f}{\lvert \nabla f \rvert}$

- **Oriented Surface:** Surface S together with a normal field (a field of unit normal vectors on S that varies continuously with position).
- **Flux** of $\vec{F} = \langle M, N, P \rangle$ across an oriented surface S in the direction of $\vec{n}$:

$$\boxed{\iint_S \vec{F} \cdot \vec{n} \, d\sigma}$$

- On a parametric surface:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_S \vec{F} \cdot \frac{\pm(\vec{r}_u \times \vec{r}_v)}{\lvert \vec{r}_u \times \vec{r}_v \rvert} d\sigma = \iint_R \vec{F} \cdot \frac{\pm(\vec{r}_u \times \vec{r}_v)}{\lvert \vec{r}_u \times \vec{r}_v\rvert} \lvert \vec{r}_u \times \vec{r}_v \rvert \, du\, dv$$

$$= \iint_R \pm\left(\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\right) du\, dv$$

- On an implicit surface:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_S \vec{F} \cdot \frac{\pm\nabla f}{\lvert\nabla f\rvert} d\sigma = \iint_R \vec{F} \cdot \frac{\pm \nabla f}{\lvert \nabla f\rvert} \frac{\lvert\nabla f\rvert}{\lvert \nabla f \cdot \vec{p}\rvert} dA$$

$$= \iint_R \pm\left(\vec{F} \cdot \frac{\nabla f}{\lvert \nabla f \cdot \vec{p} \rvert}\right) dA$$

16.7 Stokes' Theorem and 16.8 Divergence Theorem

- ∇ (del) operator: $\boxed{\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}}$

- $\vec{F} = \langle M, N, P \rangle$ vector field:

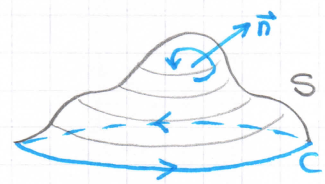
$$\boxed{\text{curl } \vec{F}} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$\boxed{\text{div } \vec{F}} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Stokes' Theorem:

Let S be a piecewise smooth oriented surface with piecewise smooth boundary curve C, and $\vec{F} = \langle M, N, P \rangle$ be a vector field with continuous first partial derivatives. Then:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma}$$



where the orientation of C is such that if the right hand thumb points in the direction of $\vec{n}$, the fingers curl in the direction of C.

Divergence Theorem:

Let S be a piecewise smooth oriented closed surface, enclosing a region D in space, and $\vec{F} = \langle M, N, P \rangle$ be a vector field with continuous first partial derivatives. Then the outward flux of $\vec{F}$ across S is given by:

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV}$$