

Name: Solutions

January 28th, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
Exam 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: _____

Problem	Possible Score	Earned Score
1	20	
2	18	
3	14	
4	17	
5	10	
6	21	
Total	100	

Remember that you must **SHOW YOUR WORK** to receive credit!

Good luck!

1. [20 pts.] Consider the points in space:

$$P(1, 2, 0); \quad Q(3, 1, 2); \quad R(-2, 0, 1).$$

a). [6 pts.] Express the vectors \vec{PQ} and \vec{PR} in standard component form.

$$\vec{PQ} = \langle 2, -1, 2 \rangle \quad (3 \text{ pts.}) - 1 \text{ pt. each}$$

$$\vec{PR} = \langle -3, -2, 1 \rangle \quad (3 \text{ pts.}) - 1 \text{ pt. each}$$

b). [7 pts.] Find:

$$\vec{PQ} \times \vec{PR}.$$

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -3 & -2 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 2 \\ -3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ -3 & -2 \end{vmatrix} \\ &= \langle 3, -8, -7 \rangle \quad (6 \text{ pts.}) \\ &\quad (1 \text{ pt.}) - \text{setting up determinant} \quad 2 \text{ pts. each} \end{aligned}$$

c). [7 pts.] Find an equation for the plane determined by the points P, Q and R . You do not need to simplify.

$$\text{Normal Vector: } \vec{PQ} \times \vec{PR} = \langle 3, -8, -7 \rangle \quad (1 \text{ pt.})$$

$$\text{Point P: } 3(x-1) - 8(y-2) - 7(z) = 0$$

$$\text{Point Q: } 3(x-3) - 8(y-1) - 7(z-2) = 0$$

$$\text{Point R: } 3(x+2) - 8(y) - 7(z-1) = 0$$

} either one (6 pts.)

2. [18 pts.] Find parametric equations for the line that is tangent to the curve:

$$\vec{r}(t) = (2\sin(t))\vec{i} + (t^4 - 5\cos(t))\vec{j} + (4e^{2t})\vec{k},$$

at the point on the curve where $t = 0$.

$$\vec{v}(t) = \langle 2\cos t, 4t^3 + 5\sin t, 8e^{2t} \rangle \quad (6\text{pts.}) - 2\text{pts. each}$$

$$\vec{v}(0) = \langle 2, 0, 8 \rangle \quad \text{vector parallel to the line} \quad (3\text{pts.})$$

$$\vec{r}(0) = \langle 0, -5, 4 \rangle \quad \text{point on the line: } (0, -5, 4) \quad (3\text{pts.})$$

$$\text{Parametric Equations:} \quad \begin{cases} x = 2t \\ y = -5 \\ z = 4 + 8t \end{cases}$$

(6pts.)
2 pts. each

3. [14 pts.] Consider the vectors:

$$\vec{u} = \langle 1, 1, 1 \rangle,$$

$$\vec{v} = \langle 2, 1, 0 \rangle.$$

a). [7 pts.] Find the dot product $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 2 + 1 = \boxed{3} \quad (7 \text{ pts.})$$

b). [7 pts.] Find the angle θ between \vec{u} and \vec{v} . Give an exact answer.

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \quad (1 \text{ pt.}) \text{ formula} \quad |\vec{u}| = \sqrt{3} \quad (2 \text{ pts.})$$

$$|\vec{v}| = \sqrt{5} \quad (2 \text{ pts.})$$

$$\cos \theta = \frac{3}{\sqrt{3} \sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \quad (1 \text{ pt.})$$

$$\boxed{\theta = \cos^{-1} \left(\frac{\sqrt{3}}{\sqrt{5}} \right)} \quad (1 \text{ pt.})$$

4. [17 pts.] Given that:

$$\frac{d\vec{r}}{dt} = (6\sqrt{t+1})\vec{i} + (e^{-t})\vec{j} + \left(\frac{1}{t+1}\right)\vec{k},$$

$$\vec{r}(0) = \vec{k},$$

find $\vec{r}(t)$.

$$\vec{r}(t) = \int \frac{d\vec{r}}{dt} \quad \text{(scribble)}$$

$$= \left\langle \int 6\sqrt{t+1} dt, \int e^{-t} dt, \int \frac{1}{t+1} dt \right\rangle$$

$$= \left\langle 6 \cdot \frac{2}{3} (t+1)^{3/2} + C_1, -e^{-t} + C_2, \ln(t+1) + C_3 \right\rangle$$

$$= \left\langle 4(t+1)^{3/2} + C_1, -e^{-t} + C_2, \ln(t+1) + C_3 \right\rangle$$

(7pts.)
(2pts.) for each component
(1pt.) for adding constants

$$\Rightarrow \vec{r}(0) = \langle 4 + C_1, -1 + C_2, C_3 \rangle$$
$$= \langle 0, 0, 1 \rangle$$

$$\left. \begin{aligned} \Rightarrow C_1 &= -4 \\ C_2 &= 1 \\ C_3 &= 1 \end{aligned} \right\}$$

(3pts.) - (1pt.) for each constant

$$\vec{r}(t) = \left\langle 4(t+1)^{3/2} - 4, -e^{-t} + 1, \ln(t+1) + 1 \right\rangle$$

5. [10 pts.] Find the length of the curve:

$$\vec{r}(t) = (t \cos(t))\vec{i} + (t \sin(t))\vec{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\vec{k},$$

from the point $(0, 0, 0)$ to the point $(-\pi, 0, \frac{2\sqrt{2}}{3}\pi^{3/2})$.

$$\vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow \textcircled{t=0} \text{ start point} \quad (1 \text{ pt.})$$

$$\vec{r}(\pi) = \langle -\pi, 0, \frac{2\sqrt{2}}{3}\pi^{3/2} \rangle \Rightarrow \textcircled{t=\pi} \text{ end point} \quad (1 \text{ pt.})$$

$$\begin{aligned} \vec{v}(t) &= \left\langle \cos t - t \sin t, \sin t + t \cos t, \frac{2\sqrt{2}}{3} \cdot \frac{3}{2} t^{1/2} \right\rangle \quad (3 \text{ pts.}) - 1 \text{ pt. each} \\ &= \langle \cos t - t \sin t, \sin t + t \cos t, \sqrt{2}\sqrt{t} \rangle \end{aligned}$$

$$\begin{aligned} |\vec{v}(t)|^2 &= \cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t \\ &\quad + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t \\ &\quad + 2t \end{aligned} \quad (1 \text{ pt.}) - \text{simplification}$$

$$= (\cos^2 t + \sin^2 t) + t^2 (\sin^2 t + \cos^2 t) + 2t \quad \left. \vphantom{= (\cos^2 t + \sin^2 t)} \right\} (1 \text{ pt.})$$

$$= 1 + 2t + t^2$$

$$= (1+t)^2$$

$$|\vec{v}(t)| = \underline{\underline{|1+t|}} = 1+t \text{ for } 0 \leq t \leq \pi \quad (1 \text{ pt.})$$

$$L = \int_0^\pi |\vec{v}(t)| dt = \int_0^\pi (1+t) dt = \left(t + \frac{t^2}{2}\right) \Big|_0^\pi = \boxed{\pi + \frac{\pi^2}{2}} \quad (1 \text{ pt.})$$

(1 pt.) - formula final answer

6. [21 pts.] Given the curve:

$$\vec{r}(t) = \langle t, 2 \sin(t), 2 \cos(t) \rangle,$$

find:

a). [8 pts.] The unit tangent vector $\vec{T}(t)$.

$$\vec{v}(t) = \langle 1, 2 \cos t, -2 \sin t \rangle \quad (3 \text{ pts.})$$

$$|\vec{v}(t)| = \sqrt{1 + 4 \cos^2 t + 4 \sin^2 t} = \sqrt{1 + 4(\sin^2 t + \cos^2 t)} = \sqrt{5} \quad (2 \text{ pts.})$$

$$\vec{T} = \left\langle \frac{1}{\sqrt{5}}, \frac{2 \cos t}{\sqrt{5}}, -\frac{2 \sin t}{\sqrt{5}} \right\rangle \quad (3 \text{ pts.})$$

$$= \frac{1}{\sqrt{5}} \langle 1, 2 \cos t, -2 \sin t \rangle$$

b). [8 pts.] The unit normal vector $\vec{N}(t)$.

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{5}} \langle 0, -2 \sin t, -2 \cos t \rangle \quad (3 \text{ pts.})$$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{1}{\sqrt{5}} \sqrt{4 \sin^2 t + 4 \cos^2 t} = \frac{2}{\sqrt{5}} \quad (2 \text{ pts.})$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \langle 0, -\sin t, -\cos t \rangle \quad (3 \text{ pts.})$$

c). [5 pts.] The unit binormal vector $\vec{B}(t)$. (You can use the back of this page for \vec{B}).

$$\vec{B} = \vec{T} \times \vec{N} \quad (1 \text{ pt.}) \text{ formula}$$
$$= \frac{1}{\sqrt{5}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2\cos t & -2\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix}$$

(1 pt.) - setting up determinant

$$= \frac{1}{\sqrt{5}} \langle -2\cos^2 t - 2\sin^2 t, \cos t, -\sin t \rangle \quad (3 \text{ pts.}) - 1 \text{ pt. each}$$

$$= \frac{1}{\sqrt{5}} \langle -2, \cos t, -\sin t \rangle$$