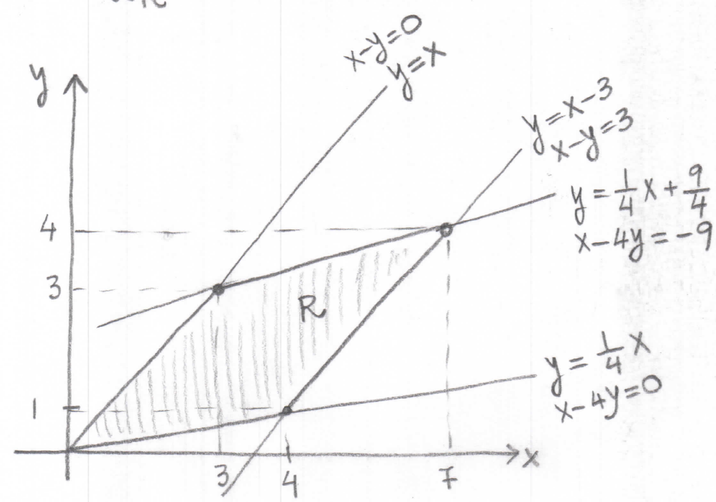
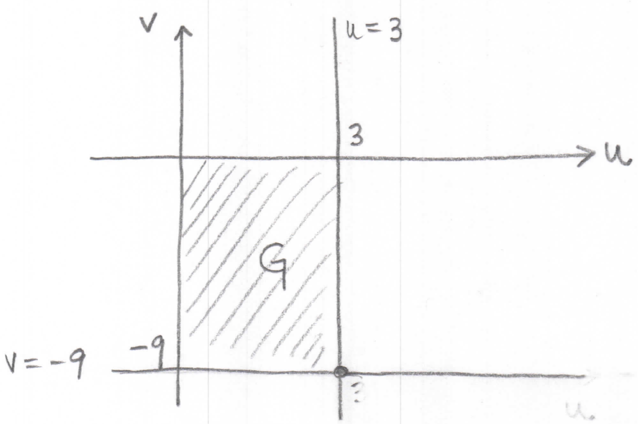
① $\iint_R e^{4x-y}\,dA$; R = parallelogram w/ vertices (0,0), (3,3), (7,4) & (4,1).



$$\boxed{\begin{aligned} u &= x-y \\ v &= x-4y \end{aligned}}$$

Boundaries of R	Boundaries of G
$x-4y=0$	$v=0$
$x-y=0$	$u=0$
$x-4y=-9$	$v=-9$
$x-y=3$	$u=3$



$$\begin{aligned} 4u &= 4x-4y \\ v &= x-4y \\ \hline 4u-v &= 3x \end{aligned}$$

$$\boxed{\begin{aligned} x &= \tfrac{1}{3}(4u-v) \\ y &= \tfrac{1}{3}(u-v) \end{aligned}}$$

$u-v=3y$

$$J = \begin{vmatrix} 4/3 & -1/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{4}{9}+\frac{1}{9} = -\frac{1}{3}$$

$$\Rightarrow \iint_R e^{4x-y}\,dA = \iint_G e^{5u-v}\,\frac{1}{3}\,d(u,v)$$

$$4x-y = \frac{4}{3}(4u-v) - \frac{1}{3}(u-v) = 5u-v$$

$$= \frac{1}{3}\int_{-9}^{0}\int_0^3 e^{5u}e^{-v}\,du\,dv$$

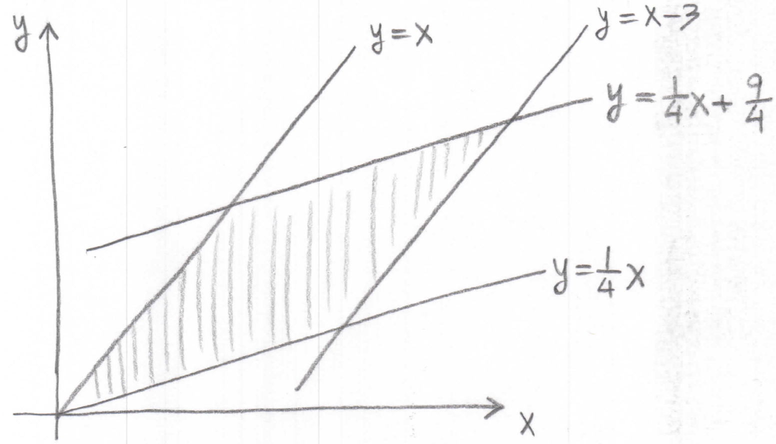
$$= \frac{1}{3}\int_{-9}^{0} \frac{1}{5}e^{5u}e^{-v}\Big|_{u=0}^{u=3}\,dv$$

$$= \frac{1}{3}\int_{-9}^{0} \frac{1}{5}(e^{15}-1)e^{-v}\,dv$$

$$= -\frac{1}{15}(e^{15}-1)\,e^{-v}\Big|_{v=-9}^{v=0}$$

$$= -\frac{1}{15}(e^{15}-1)(1-e^{9}) = \boxed{\frac{1}{15}(e^{15}-1)(e^{9}-1)}$$

Alternate Substitution $\iint_R e^{4x-y}\,dA$

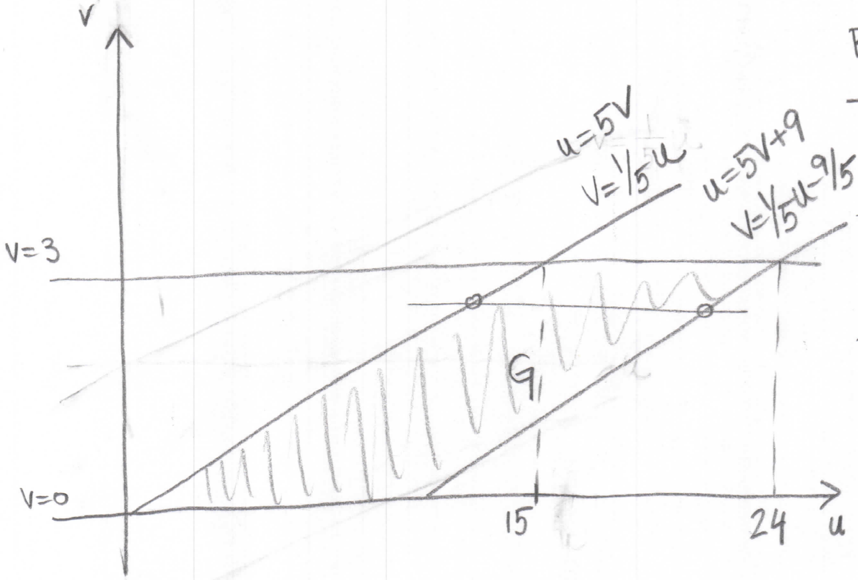


$$\begin{cases} u = 4x-y \\ v = x-y \end{cases}$$

$u - 4v = 3y$

$u - v = 3x$

$$\begin{cases} x = \frac{1}{3}(u-v) \\ y = \frac{1}{3}(u-4v) \end{cases}$$



Boundaries of R	Boundaries of G
$y = x$	$\frac{1}{3}(u-4v) = \frac{1}{3}(u-v)$ $v = 0$
$y = x-3$	$\frac{1}{3}(u-4v) = \frac{1}{3}(u-v) - 3$ $v = 3$
$y = \frac{1}{4}x + \frac{9}{4}$	$\frac{1}{3}(u-4v) = \frac{1}{12}(u-v) + \frac{9}{4}$ $v = \frac{1}{5}u - \frac{9}{5}$
$y = \frac{1}{4}x$	$\frac{1}{3}(u-4v) = \frac{1}{12}(u-v)$ $v = \frac{1}{5}u$

$$J = \begin{vmatrix} 1/3 & -1/3 \\ 1/3 & -4/3 \end{vmatrix} = -\frac{4}{9} + \frac{1}{9} = -\frac{1}{3}$$

$4x - y = u$

$$\iint_R e^{4x-y}\,dA = \int_0^3 \int_{5v}^{5v+9} e^u \cdot \frac{1}{3}\,du\,dv$$

$$= \int_0^3 \frac{1}{3} e^u \Big|_{u=5v}^{u=5v+9} dv$$

$$= \int_0^3 \frac{1}{3}\left(e^{5v+9} - e^{5v}\right) dv$$

$$= \frac{1}{3}\left(\frac{1}{5}e^{5v+9} - \frac{1}{5}e^{5v}\right)\Big|_0^3$$

$$= \frac{1}{15}\left(e^{24} - e^{15} - e^9 + 1\right)$$

$$= \frac{1}{15}\left(e^{15}(e^9-1) - (e^9-1)\right) = \boxed{\frac{1}{15}(e^{15}-1)(e^9-1)}$$

② $\iint_R (x^2+y^2)\,dA$; $R: 1 \le xy \le 4,\ 1 \le \frac{y}{x} \le 4$

$1 \le xy \le 4 \Rightarrow \frac{1}{x} \le y \le \frac{4}{x}$

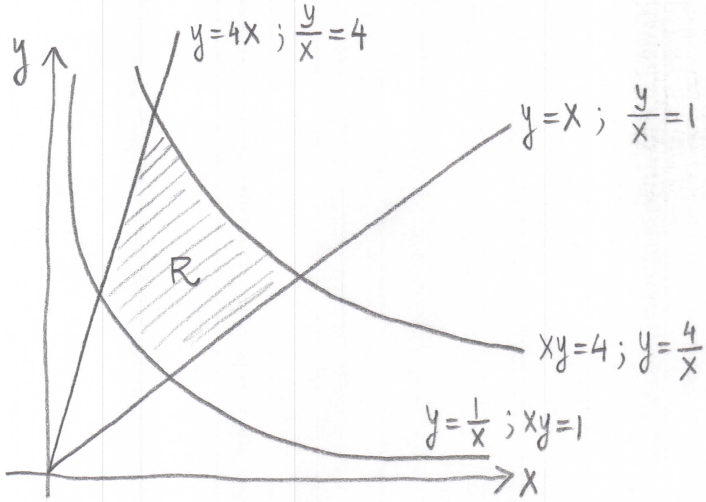
$1 \le \frac{y}{x} \le 4 \Rightarrow x \le y \le 4x$

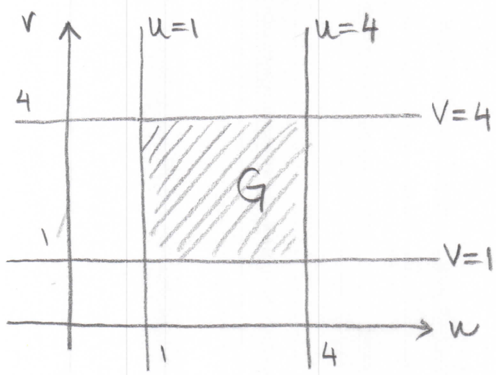
$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Rightarrow y = vx$$

$\Rightarrow u = x \cdot vx$

$\Rightarrow \frac{u}{v} = x^2 \Rightarrow x = \sqrt{\frac{u}{v}}$

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$





Boundaries of R	Boundaries of G
$\frac{y}{x}=1$	$v=1$
$xy=1$	$u=1$
$xy=4$	$u=4$
$\frac{y}{x}=4$	$v=4$

$$J = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}$$

$$\iint_R (x^2+y^2)\,dA = \iint_G \left(\frac{u}{v} + uv\right)\frac{1}{2v}\,d(u,v)$$

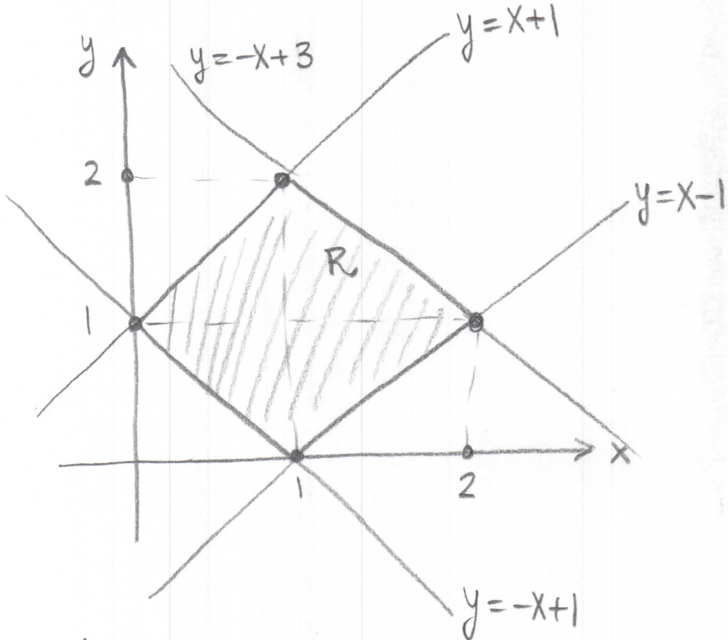
$$= \iint_G \left(\frac{u}{2v^2} + \frac{1}{2}u\right) d(u,v)$$

$$= \int_1^4\int_1^4 \frac{1}{2}\left(\frac{1}{v^2}+1\right)u\,du\,dv = \int_1^4 \frac{1}{2}\left(\frac{1}{v^2}+1\right)\frac{u^2}{2}\Bigg|_{u=1}^{u=4} dv$$

$$= \int_1^4 \frac{1}{2}\left(\frac{1}{v^2}+1\right)\left(8-\frac{1}{2}\right)dv = \frac{15}{4}\int_1^4 \left(\frac{1}{v^2}+1\right)dv$$

$$= \frac{15}{4}\left(\frac{-1}{v}+v\right)\Bigg|_1^4 = \frac{15}{4}\left(-\frac{1}{4}+4+\cancel{1}-\cancel{1}\right) = \left(\frac{15}{4}\right)^2 = \boxed{\frac{225}{16}}$$

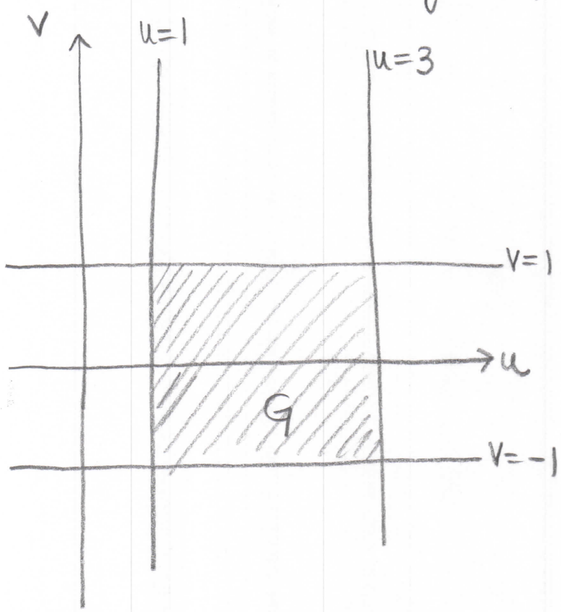
③ $\iint_R (x+y)^2 \sin^2(x-y)\,dA$; R : square w/ vertices (0,1), (1,2), (2,1), (1,0).



$$\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

Boundaries of R	Boundaries of G
$y=x+1 \Rightarrow x-y=-1$	$v=-1$
$y=x-1 \Rightarrow x-y=1$	$v=1$
$y=-x+3 \Rightarrow x+y=3$	$u=3$
$y=-x+1 \Rightarrow x+y=1$	$u=1$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



$$\iint_R (x+y)^2 \sin^2(x-y)\,dA = \iint_G u^2 \sin^2 v \, |J(u,v)| \, d(u,v)$$

$$= \int_{-1}^{1}\int_1^3 u^2 \sin^2 v \, \frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{u^3}{3}\sin^2 v \Big|_{u=1}^{u=3} dv$$

$$= \frac{1}{2}\int_{-1}^{1} \left(9\sin^2 v - \frac{1}{3}\sin^2 v\right) dv$$

$$= \frac{13}{3}\int_{-1}^{1} \sin^2 v\,dv$$

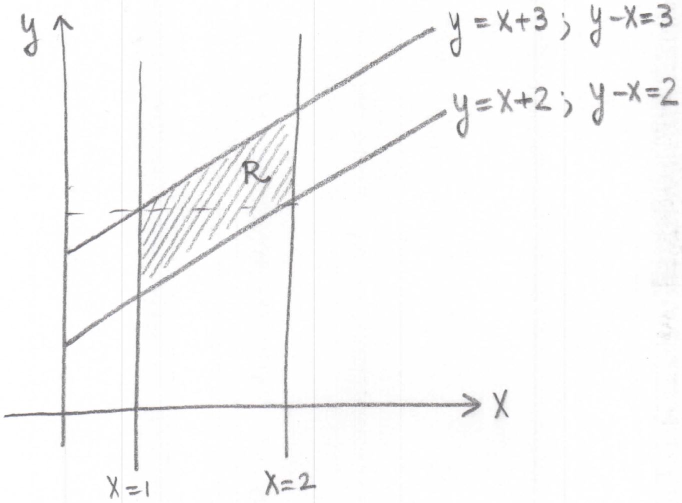
$$= \frac{13}{3}\int_{-1}^{1} \frac{1}{2}\left(1-\cos(2v)\right) dv$$

$$= \frac{13}{6}\left(v - \frac{1}{2}\sin(2v)\right)\Big|_{v=-1}^{v=1}$$

$$= \frac{13}{6}\left(1 - \frac{1}{2}\sin(2) + 1 + \frac{1}{2}\sin(-2)\right)$$

$$= \frac{13}{6}\left(2 - \frac{1}{2}\sin(2) - \frac{1}{2}\sin(2)\right) = \boxed{\frac{13}{6}\left(2-\sin(2)\right)}$$

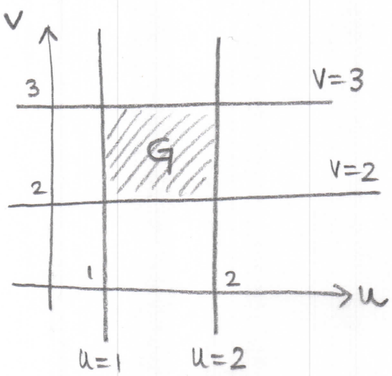
④ $I=\int_1^2\int_{x+2}^{x+3}\dfrac{dy\,dx}{\sqrt{xy-x^2}} = \int_1^2\int_{x+2}^{x+3}\dfrac{dy\,dx}{\sqrt{x(y-x)}}$



$$\begin{cases} u=x \\ v=y-x \end{cases} \Rightarrow \begin{cases} x=u \\ y=u+v \end{cases}$$

Boundaries of R	Boundaries of G
$x=1$	$u=1$
$x=2$	$u=2$
$y-x=3$	$v=3$
$y-x=2$	$v=2$

$$J=\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}=1$$



$$I=\int_1^2\int_2^3\frac{1}{\sqrt{uv}}\,dv\,du$$

$$=\int_1^2\frac{2}{\sqrt{u}}\sqrt{v}\Big|_{v=2}^{v=3}du$$

$$=\int_1^2\frac{2}{\sqrt{u}}\left(\sqrt{3}-\sqrt{2}\right)du$$

$$=4\left(\sqrt{3}-\sqrt{2}\right)\sqrt{u}\Big|_1^2 = \boxed{4\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)}$$

Compute I as was given:

$$I=\int_1^2\int_{x+2}^{x+3}\frac{1}{\sqrt{x}\sqrt{y-x}}\,dy\,dx = \int_1^2\frac{2}{\sqrt{x}}\sqrt{y-x}\Big|_{y=x+2}^{y=x+3}dx$$

$$=\int_1^2\frac{2}{\sqrt{x}}\left(\sqrt{3}-\sqrt{2}\right)dx$$

$$=4\left(\sqrt{3}-\sqrt{2}\right)\sqrt{x}\Big|_1^2$$

$$=4\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right).$$

5 $\int_0^4 \sqrt{x}\cos(\sqrt{x})\,dx$ <u>Method 1 :</u>

$u = \sqrt{x}$; $x = 0 \Rightarrow u = 0$

$du = \frac{1}{2\sqrt{x}}dx$; $x = 4 \Rightarrow u = 2$

$$\int_0^4 \sqrt{x}\cos(\sqrt{x})\,dx = \int_0^4 \underbrace{2\sqrt{x}\cdot\sqrt{x}}_{2u^2}\underbrace{\cos(\sqrt{x})}_{\cos(u)}\cdot\underbrace{\frac{1}{2\sqrt{x}}dx}_{du}$$

$$= \int_0^2 2u^2\cos(u)\,du$$

$$= \int_0^2 2u^2(\sin(u))'du = 2u^2\sin(u)\Big|_0^2 - \int_0^2 4u\sin(u)\,du$$

$$= 8\sin(2) + 4\int_0^2 u(\cos(u))'du$$

$$= 8\sin(2) + 4\left(u\cos(u)\Big|_0^2 - \int_0^2 \cos(u)\,du\right)$$

$$= 8\sin(2) + 4\left(2\cos(2) - \sin(u)\Big|_0^2\right)$$

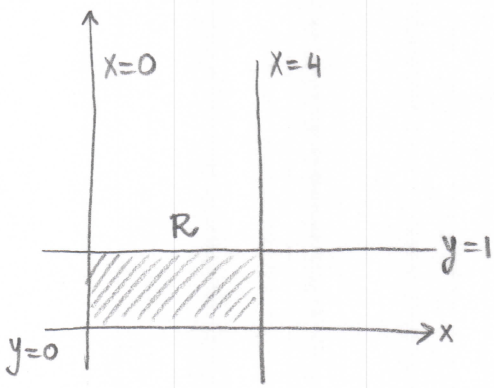
$$= 8\sin(2) + 8\cos(2) - 4\sin(2)$$

$$= \boxed{4\sin(2) + 8\cos(2)}$$

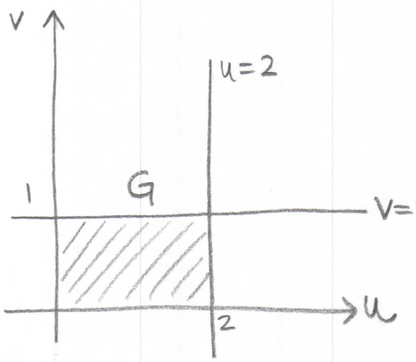
Method 2:

$$\int_0^4\int_0^1 \sqrt{x}\cos(\sqrt{x})\,dy\,dx = \int_0^4 \sqrt{x}\cos(\sqrt{x})\cdot y\Big|_{y=0}^{y=1} dx = \int_0^4 \sqrt{x}\cos(\sqrt{x})\,dx.$$

$$\begin{cases} u=\sqrt{x} \\ v=y \end{cases} \Rightarrow \begin{cases} x=u^2 \\ y=v \end{cases} \qquad J=\begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u \geq 0$$



Boundaries of R:	Boundaries of G:
$y=1$	$v=1$
$y=0$	$v=0$
$x=0$	$u^2=0 \Rightarrow u=0$
$x=4$	$u^2=4 \Rightarrow u=2$ (b/c $u\geq 0$)



$$\int_0^4 \sqrt{x}\cos(\sqrt{x})\,dx = \int_0^4\int_0^1 \sqrt{x}\cos(\sqrt{x})\,dy\,dx$$

$$= \iint_G u\cos(u)\cdot 2u \; d(u,v)$$

$$= \int_0^2\int_0^1 2u^2\cos(u)\,dv\,du$$

$$= \int_0^2 2u^2\cos(u)\cdot v\Big|_{v=0}^{v=1} du$$

$$= \boxed{\int_0^2 2u^2\cos(u)} = 4\sin(2)+8\cos(2)$$

(as before)