1. $$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

Region 1: $\sqrt{1-x^2} \le y \le x$; $\frac{1}{\sqrt{2}} \le x \le 1$

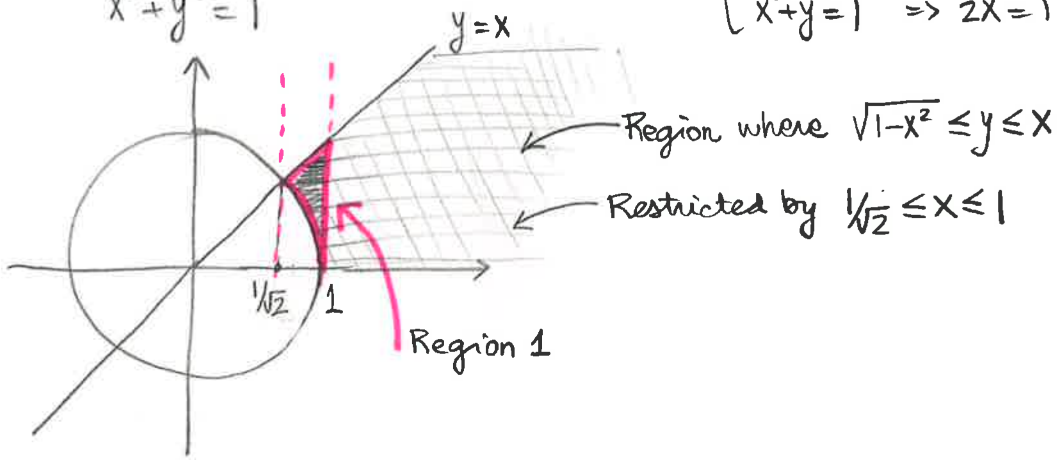
$y = \sqrt{1-x^2} \ge 0$

$y^2 = 1 - x^2$

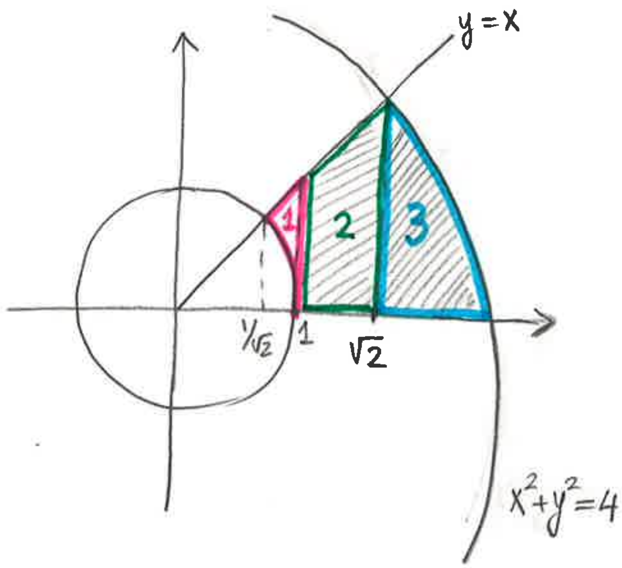
$x^2 + y^2 = 1$

Find where the line $y = x$ intersects the circle:

$$\begin{cases} y = x \\ x^2 + y^2 = 1 \end{cases} \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

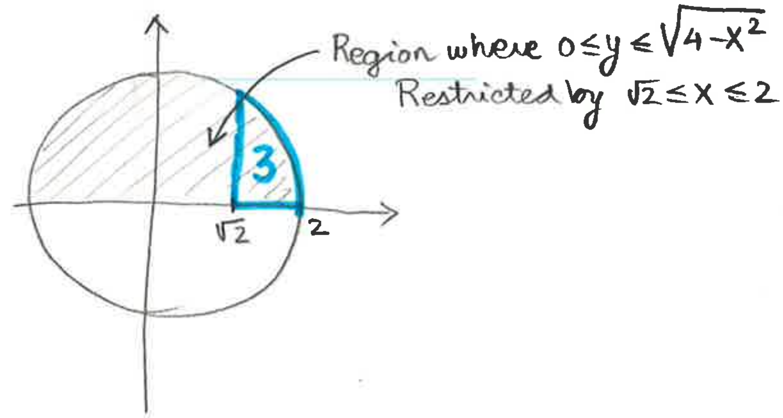


Region 2: $0 \le y \le x$; $1 \le x \le \sqrt{2}$

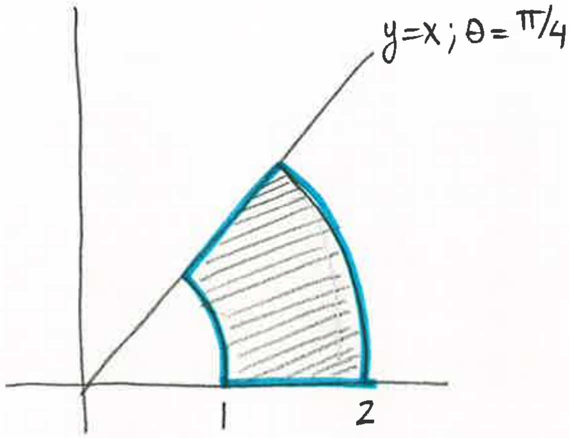


Region 3: $0 \le y \le \sqrt{4-x^2}$

$\sqrt{2} \le x \le 2$



$\Rightarrow$ Integral $\iint_R xy\,dy\,dx$, where R is the union of the three regions above



$$\iint_R xy\,dy\,dx = \int_0^{\pi/4}\int_1^2 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

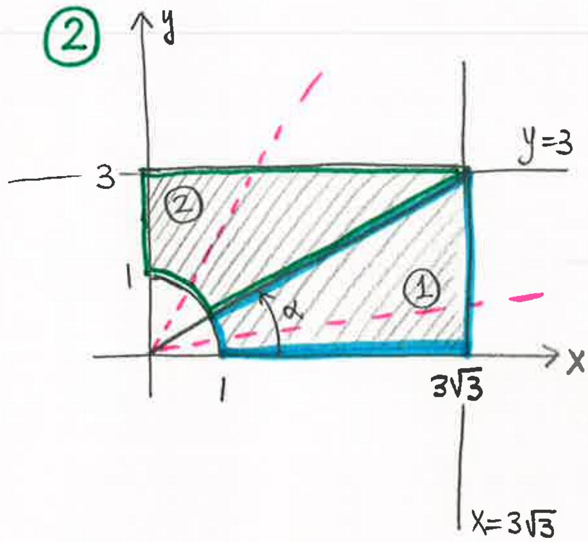
$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{r^4}{4}\cos\theta\sin\theta\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi/4} \frac{15}{4}\cos\theta\sin\theta\,d\theta$$

$$= \frac{15}{4}\,\frac{\sin^2\theta}{2}\Big|_{\theta=0}^{\pi/4} = \boxed{\frac{15}{16}}$$

2



- We have to split the region in two.
- Find α: $\tan\alpha = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\alpha = \pi/6}$
- In Region 1: $0 \le \theta \le \pi/6$
  - r is "bounded" by the vertical line $x = 3\sqrt{3}$

    $r\cos\theta = 3\sqrt{3}$

    $r = \frac{3\sqrt{3}}{\cos\theta}$

    $\Rightarrow 1 \le r \le \frac{3\sqrt{3}}{\cos\theta}$
- In Region 2: $\pi/6 \le \theta \le \pi/2$
  - r is "bounded" by the horizontal line $y=3$

    $r\sin\theta = 3$

    $r = \frac{3}{\sin\theta}$

    $\Rightarrow 1 \le r \le \frac{3}{\sin\theta}$

Describe region R in polar coordinates:

$$\boxed{\begin{cases} 0 \le \theta \le \pi/6, & 1 \le r \le \dfrac{3\sqrt{3}}{\cos\theta} \\ \pi/6 \le \theta \le \pi/2, & 1 \le r \le \dfrac{3}{\sin\theta} \end{cases}}$$

Setting up an integral over this region:

$$\int_0^{\pi/6}\int_1^{3\sqrt{3}/\cos\theta} f(r,\theta)\,r\,dr\,d\theta + \int_{\pi/6}^{\pi/2}\int_1^{3/\sin\theta} f(r,\theta)\,r\,dr\,d\theta$$

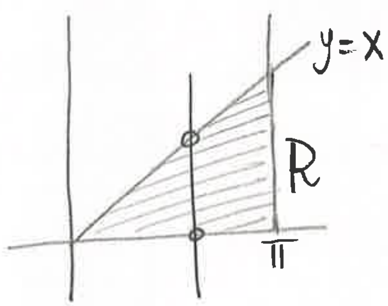
③ a). $\displaystyle\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\left(\frac{y}{x}\right) dy\,dx = \int_{\pi/2}^{\pi} \sin\left(\frac{y}{x}\right)\Big|_{y=0}^{y=x^2} dx$

$$= \int_{\pi/2}^{\pi} \sin(x)\,dx = -\cos(x)\Big|_{\pi/2}^{\pi} = -(-1-0) = \boxed{1}$$

b). $\displaystyle\int_1^4\int_0^{\sqrt{y}} e^{x/\sqrt{y}}\,dx\,dy = \int_1^4 \sqrt{y}\, e^{x/\sqrt{y}}\Big|_{x=0}^{x=\sqrt{y}} dy$

$$= \int_1^4 \sqrt{y}(e-1)\,dy = \frac{2}{3}y^{3/2}(e-1)\Big|_1^4 = \frac{2}{3}(e-1)(8-1) = \boxed{\frac{14}{3}(e-1)}$$

c) $\displaystyle\iint_R \frac{\sin x}{x}\,dA$; $R = \{(x,y) : 0 \le y \le x,\ 0 \le x \le \pi\}$



Vertical cross-sections (because we cannot integrate $\frac{\sin x}{x}$ with respect to x)

$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\,dy\,dx = \int_0^{\pi} \frac{\sin x}{x}\cdot y\Big|_{y=0}^{y=x} dx$$

$$= \int_0^{\pi} \sin x\,dx = -\cos(x)\Big|_0^{\pi}$$

$$= -(-1-1) = \boxed{2}$$

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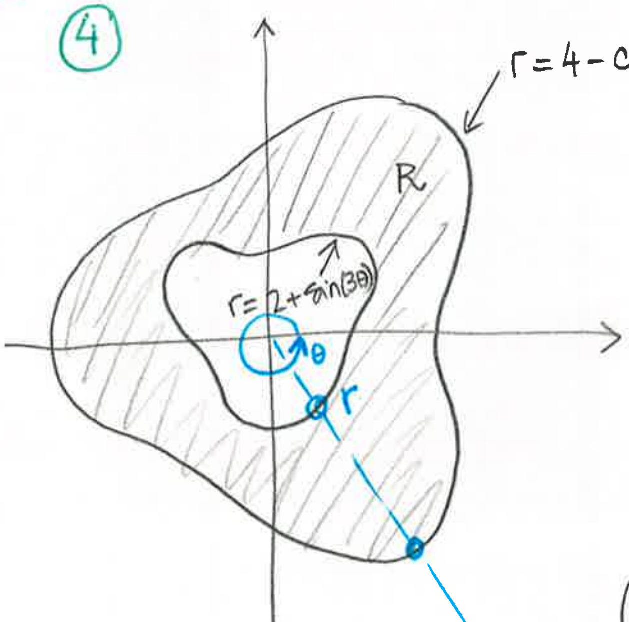
④



$$\text{Area} = \int_0^{2\pi}\int_{2+\sin(3\theta)}^{4-\cos(3\theta)} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=2+\sin(3\theta)}^{r=4-\cos(3\theta)} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left((4-\cos(3\theta))^2 - (2+\sin(3\theta))^2\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(16 - 8\cos(3\theta) + \cos^2(3\theta) - 4 - 4\sin(3\theta) - \sin^2(3\theta)\right) d\theta$$

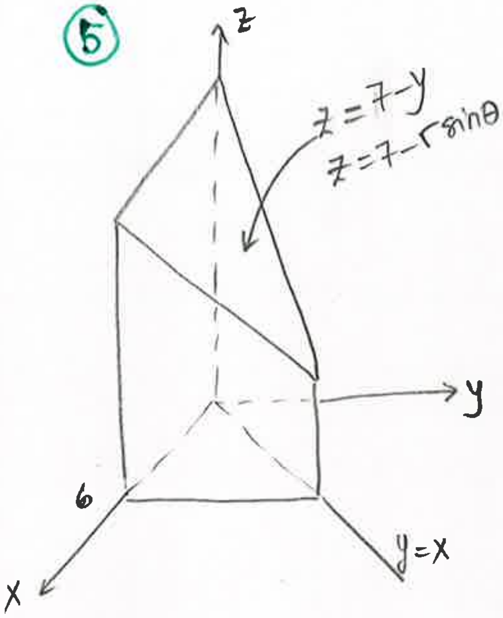
$(\cos(2\theta) = \cos^2\theta - \sin^2\theta)$

$$= \frac{1}{2}\int_0^{2\pi} (12 - 8\cos(3\theta) - 4\sin(3\theta) + \cos(6\theta))\,d\theta$$

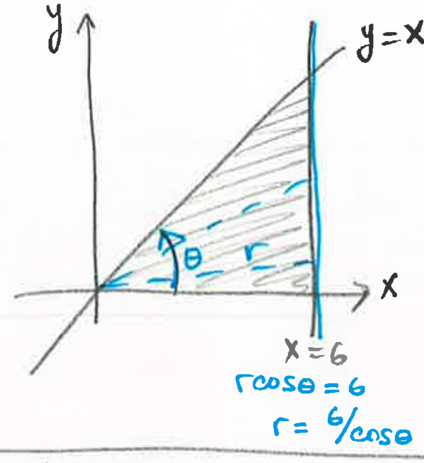
$$= \frac{1}{2}\left(12\theta - \frac{8}{3}\sin(3\theta) + \frac{4}{3}\cos(3\theta) + \frac{1}{6}\sin(6\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\,12\cdot 2\pi = \boxed{12\pi}$$

(5)



<u>Base</u> :



$$\boxed{\text{Vol} = \int_0^{\pi/4}\int_0^{\frac{6}{\cos\theta}}\int_0^{7-r\sin\theta} r\,dz\,dr\,d\theta}$$

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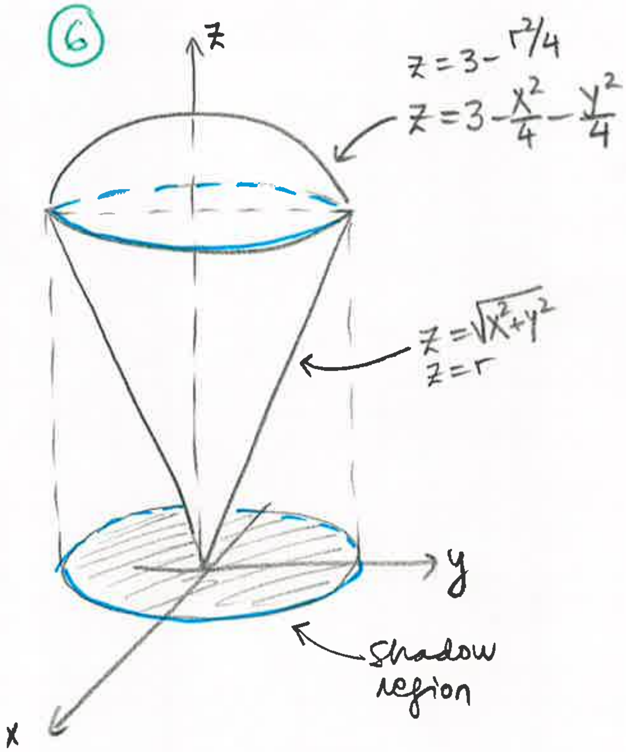
(6)



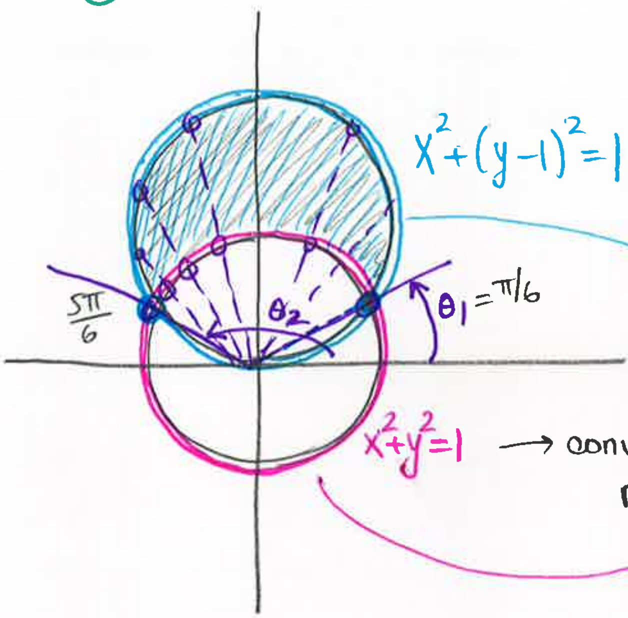
Find radius of the circle of intersection of the two surfaces: Set $z = 3 - r^2/4$ and $z = r$ equal to each other (easier to do in polar)

$$3 - \frac{r^2}{4} = r$$
$$12 - r^2 = 4r$$
$$r^2 + 4r - 12 = 0$$
$$(r+6)(r-2) = 0 \Rightarrow \boxed{r = 2}$$

So the shadow region is a disc of radius 2, centered @ the origin :

$$\text{Vol} = \int_0^{2\pi}\int_0^2\int_r^{3-r^2/4} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 rz\Big|_{z=r}^{z=3-r^2/4} dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 r\left(3 - \frac{r^2}{4} - r\right)dr\,d\theta = \int_0^{2\pi}\int_0^2\left(3r - \frac{r^3}{4} - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{3r^2}{2} - \frac{r^4}{16} - \frac{r^3}{3}\right)\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi}\frac{7}{3}\,d\theta = \frac{7}{3}\theta\Big|_0^{2\pi} = \boxed{\frac{14\pi}{3}}$$

7) Area inside $x^2+(y-1)^2=1$, outside $x^2+y^2=1$.



$x^2+(y-1)^2=1 \longrightarrow$ convert to polar:

$$(r\cos\theta)^2+(r\sin\theta-1)^2=1$$

$$\underbrace{r^2\cos^2\theta+r^2\sin^2\theta}_{r^2}-2r\sin\theta+\cancel{1}=\cancel{1}$$

$$r^2-2r\sin\theta=0$$

$$r(r-2\sin\theta)=0$$

$$\boxed{r=2\sin\theta}$$

$x^2+y^2=1 \longrightarrow$ convert to polar:

$$r=1 \quad \boxed{r=1}$$

To find the bounds for θ, we must find the angles θ_1 & θ_2 where the circles intersect : set the two equations $r=1$ & $r=2\sin\theta$ equal to each other:

$$1=2\sin\theta \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6},\ \frac{5\pi}{6}$$

$$\underline{\text{Area}} = \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\Big|_{r=1}^{r=2\sin\theta} d\theta$$

$$= \int_{\pi/6}^{5\pi/6}\left(2\sin^2\theta-\frac{1}{2}\right)d\theta$$

$$= \int_{\pi/6}^{5\pi/6}\left(\frac{1}{2}-\cos(2\theta)\right)d\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \left(\frac{1}{2}\theta-\frac{1}{2}\sin(2\theta)\right)\Big|_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)-\frac{1}{2}\left(\sin\frac{5\pi}{3}-\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{2\pi}{3}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$$