1. $\int_S y\,d\sigma$; $S: z = x + y^2,\ 0 \le x \le 1,\ 0 \le y \le 2.$

Method 1: Level Surface

- $S: f(x,y,z) = x + y^2 - z = 0$

  $\nabla f = \langle 1, 2y, -1 \rangle$

- Shadow Region $R: 0 \le x \le 1,\ 0 \le y \le 2$

  xy plane $\Rightarrow \vec{p} = \vec{k}$

  $\nabla f \cdot \vec{k} = -1$

- Surface Differential: $d\sigma = \dfrac{|\nabla f|\,dA}{|\nabla f \cdot \vec{k}|} = \dfrac{\sqrt{2+4y^2}}{1}\,dA$

$$\Rightarrow \int_S y\,d\sigma = \iint_R y\sqrt{2+4y^2}\,dA$$

$$= \int_0^1 \int_0^2 y\sqrt{2+4y^2}\,dy\,dx$$

$$= \frac{1}{8} \cdot \frac{2}{3} (2+4y^2)^{3/2} \Big|_0^2$$

$$= \frac{1}{12}\left(18\sqrt{18} - 2\sqrt{2}\right) = \frac{\sqrt{2}}{6}(27-1) = \boxed{\frac{13\sqrt{2}}{3}}$$

Method 2: Parametrize Surface: $x = u;\ y = v \Rightarrow z = u + v^2$

$S: \vec{r}(u,v) = \langle u, v, u+v^2 \rangle;\ u \in [0,1],\ v \in [0,2].$

$$\left.\begin{aligned} \vec{r}_u &= \langle 1, 0, 1 \rangle \\ \vec{r}_v &= \langle 0, 1, 2v \rangle \end{aligned}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \langle -1, -2v, 1 \rangle$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{2+4v^2}$$

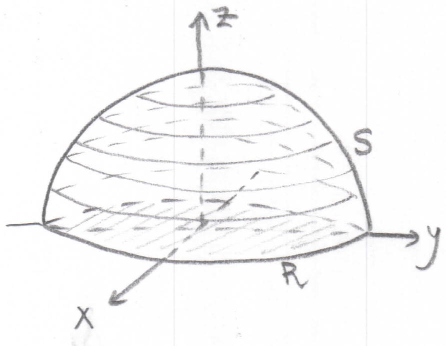
$\Rightarrow$ Surface Differential: $d\sigma = \sqrt{2+4v^2}\,d(u,v)$

$G(x,y,z) = y \Rightarrow G(\vec{r}(u,v)) = v$

$$\Rightarrow \int_S y\,d\sigma = \int_0^1 \int_0^2 v\sqrt{2+4v^2}\,dv\,du = \frac{13\sqrt{2}}{3}$$

(Same integral as above)

② $\int_S (x^2 z + y^2 z)\, d\sigma$; S: $x^2+y^2+z^2=4$, $z \geq 0$



Level Surface: $f(x,y,z) = x^2+y^2+z^2 = 4$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$|\nabla f| = 2\underbrace{\sqrt{x^2+y^2+z^2}}_{=4} = \underline{\underline{4}}$

Shadow Region R : $x^2+y^2 \leq 4$

xy plane $\Rightarrow \vec{p} = \vec{k}$

$\nabla f \cdot \vec{p} = \nabla f \cdot \vec{k} = 2z$

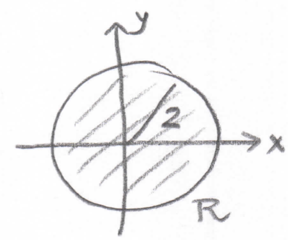
Surface Differential: $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \vec{k}|} = \dfrac{4}{|2z|} = \dfrac{4}{2z} = \dfrac{2}{z}$ (because $z > 0$)

$$\Rightarrow \int_S (x^2 z + y^2 z)\, d\sigma = \iint_R (x^2+y^2) z \cdot \frac{2}{z}\, dA$$

$$= \iint_R 2(x^2+y^2)\, dA$$

$$= \int_0^{2\pi}\int_0^2 2r^2 \cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{r^4}{2}\Big|_0^2 = \boxed{16\pi}$$



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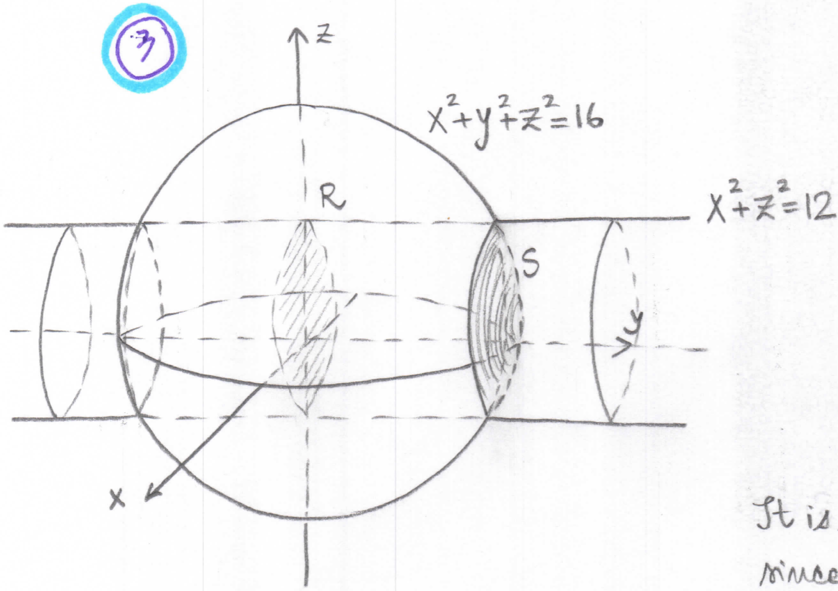
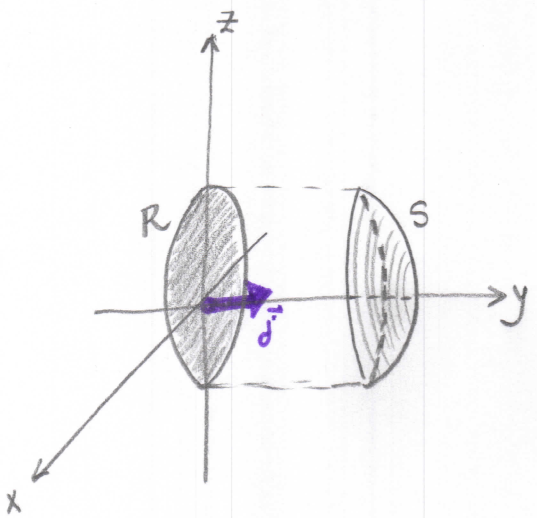


It is easier to find the surface area of one of the "spherical caps" S and subtract from the surface area of the whole sphere, which is

$$4\pi R^2 = 4\pi \cdot 16 = 64\pi.$$



It is easy to express S as a level surface since we know S is part of a sphere:

$$f(x,y,z) = x^2+y^2+z^2 = 16$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\Rightarrow |\nabla f| = 2\sqrt{x^2+y^2+z^2} = \underline{8}$$

Since the shadow region is in the xz-plane

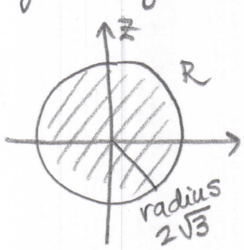
$$\vec{p} = \vec{j}$$

$$\Rightarrow \nabla f \cdot \vec{p} = \nabla f \cdot \vec{j} = 2y$$

$$\Rightarrow |\nabla f \cdot \vec{p}| = |2y| = 2y \quad (\text{since } y > 0 \text{ on } S)$$

Surface Differential: $d\sigma = \dfrac{8\,dA}{2y} = \dfrac{4}{y}\,dA$

We need to figure out what R is. But R is just "cut" into the xz-plane by the cylinder, so R is given by $x^2+z^2=12$. So:



$$\text{Surf. Area}(S) = \int_S d\sigma = \iint_R \underline{\underline{\frac{4}{y}\,dA}}$$

integration in the xz-plane so we must express y in terms of x and z; we do this by recalling that S is part of the sphere, so any point (x,y,z) on S satisfies: $x^2+y^2+z^2=16$. Since y is positive on S:

$$y = \sqrt{16-(x^2+z^2)}$$

$$= \iint_R \frac{4}{\sqrt{16-(x^2+z^2)}}\,dA$$

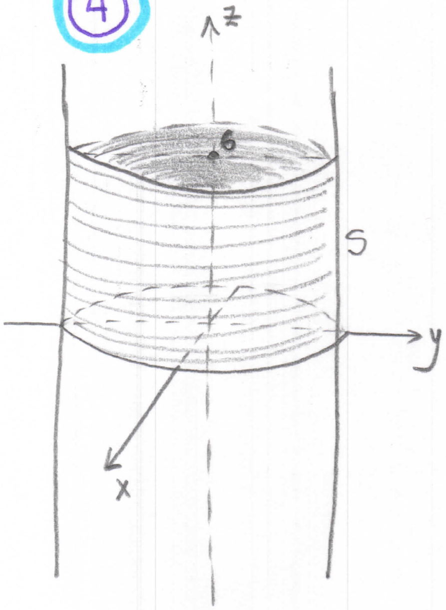
$$= \int_0^{2\pi}\int_0^{2\sqrt{3}} \frac{4}{\sqrt{16-r^2}}\, r\,dr\,d\theta$$

$$= -8\pi\sqrt{16-r^2}\,\Big|_0^{2\sqrt{3}}$$

$$= -8\pi\left(\sqrt{4}-\sqrt{16}\right) = \boxed{16\pi}$$

So the surface area of what is left of the sphere once we cut out the spherical caps is $64\pi - 2\cdot 16\pi = \boxed{32\pi}$

4



$\int_S y \, d\sigma$; S: portion of $x^2+y^2=3$ b/w $z=0$, $z=6$.

Parametrize Cylinder: $S: \vec{r}(\theta, z) = \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, z \rangle$

$0 \le \theta \le 2\pi$; $0 \le z \le 6$

$$\vec{r}_\theta = \langle -\sqrt{3}\sin\theta, \sqrt{3}\cos\theta, 0 \rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, 0 \rangle$$

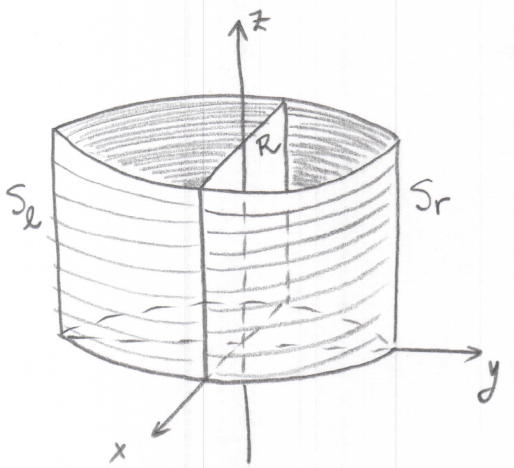
$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{3} \quad \Rightarrow \quad d\sigma = \sqrt{3}\, d(\theta, z)$$

$$G(x,y,z) = y \Rightarrow G(\vec{r}(\theta,z)) = \sqrt{3}\sin\theta$$

$$\int_S y\, d\sigma = \int_0^6 \int_0^{2\pi} \sqrt{3}\sin\theta \cdot \sqrt{3}\, d\theta\, dz$$

$$= 6 \cdot (-3\cos\theta)\Big|_0^{2\pi} = \boxed{0}.$$

Or: Take Shadow Regions. If we split S into its left and right halves (as it intersects the x-axis), note that the function we are integrating $G(x,y,z)=y$ is symmetric, i.e.



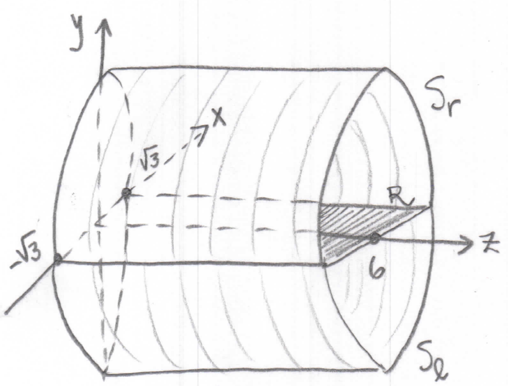
$$\int_{S_\ell} y\, d\sigma = -\int_{S_r} y\, d\sigma$$

So $\int_S y\, d\sigma = \int_{S_r} y\, d\sigma + \int_{S_\ell} y\, d\sigma$

$$= \int_{S_r} y\, d\sigma - \int_{S_r} y\, d\sigma = \boxed{0}.$$

Had we not observed that, we could still just compute the integral, splitting into $\int_{S_r}$ and $\int_{S_\ell}$ and taking the shadow region R: $-\sqrt{3} \le x \le \sqrt{3}$; $0 \le z \le 6$

In both cases, the surfaces satisfy $x^2+y^2=3$, and $\vec{p} = \vec{j}$ (because R lies in the xz-plane).



$S_r : f(x,y,z) = x^2+y^2=3$

$\nabla f = \langle 2x, 2y, 0\rangle$

$|\nabla f| = 2\sqrt{\underbrace{x^2+y^2}_{3}} = 2\sqrt{3}$

$\vec{p} = \vec{j}$; $\nabla f \cdot \vec{p} = \nabla f \cdot \vec{j} = 2y$

$|\nabla f \cdot \vec{p}| = |2y| = 2y$

(because $y \geq 0$ on S_r!)

$$\int_{S_r} y\, d\sigma = \iint_R \cancel{y} \cdot \frac{2\sqrt{3}}{2\cancel{y}}\, dA$$

$= \sqrt{3}\ \text{Area}(R)$

$= \sqrt{3}\,(2\sqrt{3}\cdot 6)$

$= 36$

$S_\ell : f(x,y,z) = x^2+y^2=3$

$|\nabla f| = 2\sqrt{3}$

$\vec{p} = \vec{j}$; $\nabla f \cdot \vec{p} = \nabla f \cdot \vec{j} = 2y$

$|\nabla f \cdot \vec{p}| = |2y| = -2y$

because $y \leq 0$ on S_ℓ!

$$\int_{S_\ell} y\, d\sigma = \iint_R y \cdot \frac{2\sqrt{3}}{-2y}\, dA$$

$= -\sqrt{3}\ \text{Area}(R)$

$= -36$

$\Rightarrow \int_S y\, d\sigma = 36 - 36 = \boxed{0}.$

5 $\int_S x^2yz\,d\sigma$ S: part of the plane $z=1+2x+3y$ that lies above $R: 0\le x\le 3,\ 0\le y\le 2.$

$f(x,y,z)=2x+3y-z+1=0$

$\nabla f=\langle 2,3,-1\rangle$

$|\nabla f|=\sqrt{14}$

R in xy-plane $\Rightarrow \vec{p}=\vec{k}$

$\nabla f\cdot\vec{p}=\nabla f\cdot\vec{k}=-1$

$|\nabla f\cdot\vec{p}|=1$

$$d\sigma=\sqrt{14}\,dA$$

$$\int_S x^2yz\,d\sigma=\iint_R x^2yz\sqrt{14}\,dA$$

$$=\int_0^3\int_0^2 x^2yz\sqrt{14}\,dy\,dx$$

$$=\sqrt{14}\int_0^3\int_0^2 x^2y(1+2x+3y)\,dy\,dx$$

$$=\sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx$$

$$=\sqrt{14}\int_0^3\left(\frac{x^2y^2}{2}+x^3y^2+x^2y^3\right)\Bigg|_{y=0}^{y=2}dx$$

$$=\sqrt{14}\int_0^3(2x^2+4x^3+8x^2)\,dx=\left(\frac{10}{3}x^3+x^4\right)\Bigg|_0^3\cdot\sqrt{14}=\boxed{171\sqrt{14}}$$

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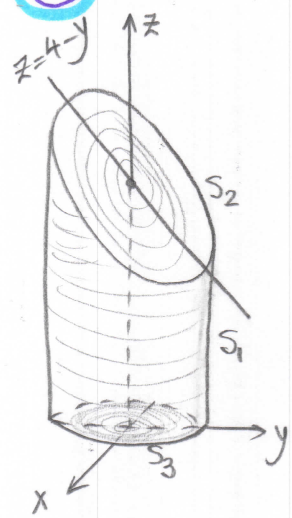


$\int_S (y+z)\,d\sigma$ S: sides: $\underbrace{\text{cylinder } x^2+y^2=3}_{S_1}$; top: $\underbrace{\text{plane } z=4-y}_{S_2}$

bottom: $\underbrace{\text{disc } x^2+y^2\le 3}_{S_3}$

$$\int_S (y+z)\,d\sigma = \int_{S_1}(y+z)\,d\sigma + \int_{S_2}(y+z)\,d\sigma + \int_{S_3}(y+z)\,d\sigma$$

S_1 : $d\sigma = \sqrt{3}\,d(\theta, z) \rightsquigarrow$ already computed in #4

S_1: $\vec{r}(\theta, z) = \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, z\rangle$;

$0 \le \theta \le 2\pi$; $0 \le z \le 4-y = \underline{4-\sqrt{3}\sin\theta}$

(the cylinder here is not bounded above by a flat plane as in #4, but by $z=4-y$).

$$\Rightarrow \int_{S_1}(y+z)\,d\sigma = \int_0^{2\pi}\int_0^{4-\sqrt{3}\sin\theta} (\sqrt{3}\sin\theta + z)\sqrt{3}\,dz\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left[(\sqrt{3}\sin\theta)z + \frac{z^2}{2}\right]\Bigg|_{z=0}^{z=4-\sqrt{3}\sin\theta} d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left[\cancel{4\sqrt{3}\sin\theta} - 3\sin^2\theta + \frac{1}{2}(16 - \cancel{8\sqrt{3}\sin\theta} + 3\sin^2\theta)\right] d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(-\frac{3}{2}\sin^2\theta + 8\right)d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(8 - \frac{3}{4}(1-\cos(2\theta))\right)d\theta$$

$$= \sqrt{3}\cdot\left(\frac{29}{4}\theta + \frac{3}{8}\sin(2\theta)\right)\Bigg|_0^{2\pi} = \boxed{\frac{29\sqrt{3}}{2}\pi}$$

S_2: $\int_{S_2}(y+z)\,d\sigma = \int_{S_2}(y+(4-y))\,d\sigma = 4\int_{S_2} d\sigma$ because on S_2, $z=4-y$.

S_2 : $f(x,y,z) = y+z = 4$

$\nabla f = \langle 0,1,1\rangle$; $|\nabla f| = \sqrt{2}$

$\vec{p} = \vec{k}$; $\nabla f\cdot\vec{k} = 1$

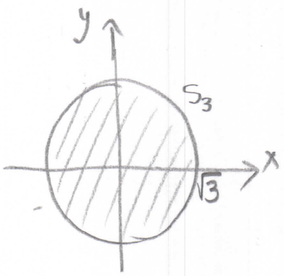
(Shadow region for S_2 is actually $R = S_3$)

$= 4\iint_{S_3}\sqrt{2}\,dA = 4\sqrt{2}\,\text{Area}(S_3)$

$= 4\sqrt{2}\,\pi\cdot(\sqrt{3})^2$

$= \boxed{12\sqrt{2}\,\pi}$

$\textcircled{S_3}$: $x^2+y^2 \le 3$



$$\int_{S_3} (y+z)\,d\sigma = \iint_{S_3} y\,dA$$

because $z=0$ on S_3

because S_3 is a "flat" surface lying in the xy-plane so $d\sigma$ is just dA

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (r\sin\theta)\, r\,dr\,d\theta$$

$$= \frac{r^3}{3}\Big|_0^{\sqrt{3}} \cdot \underbrace{(-\cos\theta)\Big|_0^{2\pi}}_{0} = \boxed{0}$$

$$\Rightarrow \int_S (y+z)\,d\sigma = \boxed{\left(\frac{29\sqrt{3}}{2} + 12\sqrt{2}\right)\pi}$$

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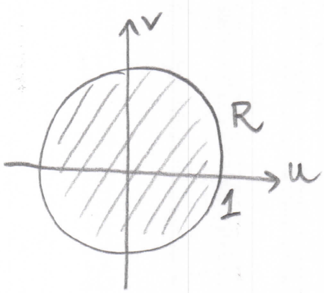
⑦ $\int_S yz\,d\sigma$; $S: \vec{r}(u,v) = \langle uv, u+v, u-v\rangle$; $u^2+v^2 \le 1$.

$$\left.\begin{aligned}\vec{r}_u &= \langle v, 1, 1\rangle \\ \vec{r}_v &= \langle u, 1, -1\rangle\end{aligned}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \langle -2, u+v, v-u\rangle$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{4 + u^2 + \cancel{2uv} + v^2 + u^2 - \cancel{2uv} + v^2} = \sqrt{4+2(u^2+v^2)}$$



$$\int_S yz\,d\sigma = \iint_R (u+v)(u-v)\sqrt{4+2(u^2+v^2)}\,dA$$

$$= \iint_R (u^2-v^2)\sqrt{4+2(u^2+v^2)}\,dA$$

$$= \int_0^{2\pi}\int_0^1 r^2\underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos(2\theta)}\sqrt{4+2r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \cos(2\theta)\, r^3\sqrt{4+2r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos(2\theta) \int_0^1 r^3\sqrt{4+2r^2}\,dr\,d\theta$$

$$= \underbrace{\left(\int_0^{2\pi}\cos(2\theta)\,d\theta\right)}_{=\frac{1}{2}\sin(2\theta)\big|_0^{2\pi} = \underline{\underline{0}}}\underbrace{\left(\int_0^1 r^3\sqrt{4+2r^2}\,dr\right)}_{\text{some \#}} = \boxed{0}$$