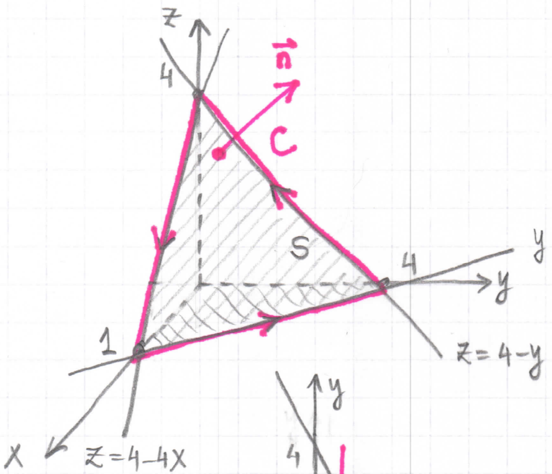
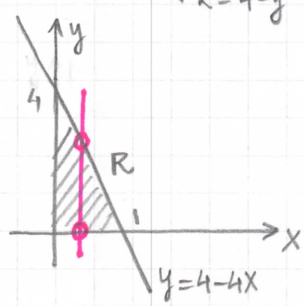
① $\vec{F} = \langle y, xz, x^2 \rangle$

C: boundary of triangle cut from $4x+y+z=4$ by the first octant (counterclockwise when viewed from above)

$\oint_C \vec{F}\cdot d\vec{r} = ?$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & x^2 \end{vmatrix} = \langle -x, -2x, z-1 \rangle$$




Stokes: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F})\cdot \vec{n}\, d\sigma$

$S$ = triangle bounded by C

$S$ = implicit surface $f(x,y,z)=4$,

$f = 4x+y+z$.

So $\nabla f = \langle 4,1,1 \rangle$ and $|\nabla f| = \sqrt{18}$.

Outer normal to S: $\vec{n} = \dfrac{\pm \nabla f}{|\nabla f|} = \pm \dfrac{1}{\sqrt{18}}\langle 4,1,1\rangle$

$$\boxed{\vec{n} = \frac{1}{\sqrt{18}}\langle 4,1,1\rangle.}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \frac{1}{\sqrt{18}}(-4x-2x+z-1)\,d\sigma$$

$$= \iint_R \frac{1}{\sqrt{18}}(-6x+z-1)\frac{|\nabla f|}{|\nabla f\cdot \vec{k}|}\,dA = \iint_R \frac{1}{\sqrt{18}}(-6x+z-1)\frac{\sqrt{18}}{1}\,dA$$

$$= \iint_{R} (-6x+\underline{\underline{z}}-1)\,dA = \iint_R \Big(-6x+\underbrace{(4-4x-y)}_{\text{from } f=4}-1\Big)\,dA$$

xy-plane

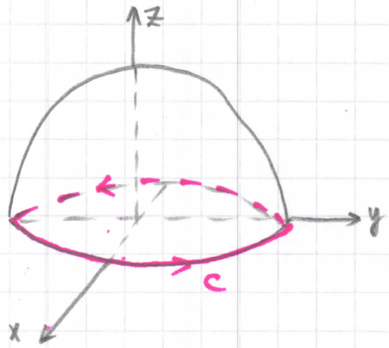
$$= \iint_R (-10x-y+3)\,dA = \int_0^1\int_0^{4-4x}(-10x-y+3)\,dy\,dx$$

$$= \int_0^1 \left(-10xy - \frac{y^2}{2}+3y\right)\Bigg|_{y=0}^{y=4-4x} dx = \int_0^1 \left(-10x(4-4x) - \frac{(4-4x)^2}{2} + 3(4-4x)\right)dx$$

$$= 4\int_0^1 (-10x+10x^2-2+4x-2x^2+3-3x)\,dx = 4\int_0^1 (8x^2-9x+1)\,dx$$

$$= 4\left(\frac{8x^3}{3} - \frac{9x^2}{2}+x\right)\Bigg|_0^1 = 4\left(\frac{8}{3}-\frac{9}{2}+1\right) = 4\cdot\frac{-5}{6} = \boxed{-\frac{10}{3}}$$

② $\iint_S \nabla \times (3y\vec{i}) \cdot \vec{n}\, d\sigma$, S: $x^2+y^2+z^2=1$, $z \geq 0$.




$\vec{F} = \langle 3y, 0, 0 \rangle$

$$\iint_S \nabla \times (3y\vec{i}) \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

C: $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$

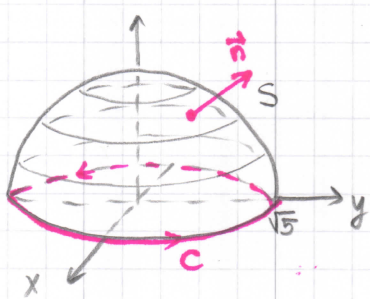
$d\vec{r}(t) = \langle -\sin t, \cos t, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 3\sin t, 0, 0 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -3\sin^2 t\, dt = -\frac{3}{2}\left(t - \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{-3\pi}$$

③ $\vec{F} = \langle 4y,\ 5-5x,\ z^2-2 \rangle$

S: $\vec{r}(\phi, \theta) = \langle \sqrt{5}\sin\phi\cos\theta,\ \sqrt{5}\sin\phi\sin\theta,\ \sqrt{5}\cos\phi \rangle$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$




S: upper hemisphere of sphere centered at O w/radius $\sqrt{5}$.

C: $\vec{r}(\theta) = \langle \sqrt{5}\cos\theta, \sqrt{5}\sin\theta, 0 \rangle$, $0 \leq \theta \leq 2\pi$

$d\vec{r}(\theta) = \langle -\sqrt{5}\sin\theta, \sqrt{5}\cos\theta, 0 \rangle$

$\vec{F}(\vec{r}(\theta)) = \langle 4\sqrt{5}\sin\theta, 5 - 5\sqrt{5}\cos\theta, -2 \rangle$

$$\iint_S (\text{curl}\,\vec{F}) \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-20\sin^2\theta + 5\sqrt{5}\cos\theta - 25\cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(-20 + 5\sqrt{5}\cos\theta - 5\cos^2\theta\right) d\theta$$

$$= \left(-20\theta + 5\sqrt{5}\sin\theta - \frac{5}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right)\Big|_0^{2\pi}$$

$$= -40\pi - \frac{5}{2} \cdot 2\pi$$

$$= \boxed{-45\pi}$$