1 a) $f(x,y) = xy^3 + x^2y^2$

$$\frac{\partial f}{\partial x} = y^3 + 2xy^2 \ ; \ \frac{\partial f}{\partial y} = 3xy^2 + 2x^2y$$

b) $f(x,y) = xe^{2x+3y}$

$$\frac{\partial f}{\partial x} = e^{2x+3y} + 2xe^{2x+3y} \ ; \ \frac{\partial f}{\partial y} = 3xe^{2x+3y}$$

c) $f(x,y) = \dfrac{x-y}{x+y}$

$$\frac{\partial f}{\partial x} = \frac{1(x+y) - 1(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \ ; \ \frac{\partial f}{\partial y} = \frac{-1(x+y) - 1(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

d) $f(x,y) = 2x \sin(x^2y)$

$$\frac{\partial f}{\partial x} = 2\sin(x^2y) + 4x^2y\cos(x^2y) \ ; \ \frac{\partial f}{\partial y} = 2x^3\cos(x^2y)$$

e) $f(x,y,z) = x\cos z + x^2y^3e^z$

$$\frac{\partial f}{\partial x} = \cos z + 2xy^3e^z \ ; \ \frac{\partial f}{\partial y} = 3x^2y^2e^z \ ; \ \frac{\partial f}{\partial z} = x^2y^3e^z$$

2 $u(x,y) = \ln(1+xy^2)$

$$\frac{\partial u}{\partial x} = \frac{y^2}{1+xy^2} \ ; \ \frac{\partial^2 u}{\partial x^2} = -\frac{y^2}{(1+xy^2)^2}\cdot y^2 = -\frac{y^4}{(1+xy^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{2y(1+xy^2) - y^2(2xy)}{(1+xy^2)^2} = \frac{2y + 2xy^3 - 2xy^3}{(1+xy^2)^2} = \frac{2y}{(1+xy^2)^2}$$

$$\Rightarrow 2\frac{\partial^2 u}{\partial x^2} + y^3\frac{\partial^2 u}{\partial y \partial x} = -\frac{2y^4}{(1+xy^2)^2} + \frac{2y^4}{(1+xy^2)^2} = \underline{\underline{0}}$$

3) $g(s,t) = f(s^2-t^2,\ t^2-s^2)$

Let $x = s^2-t^2$ and $y = t^2-s^2$. Then $g(s,t) = f(x,y)$, and by the Chain Rule:

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{\partial f}{\partial x}\right)(2s) + \left(\frac{\partial f}{\partial y}\right)(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{\partial f}{\partial x}\right)(-2t) + \left(\frac{\partial f}{\partial y}\right)(2t)$$

$$\Rightarrow t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \cancel{\left(\frac{\partial f}{\partial x}\right)(2st)} + \cancel{\left(\frac{\partial f}{\partial y}\right)(-2st)} + \cancel{\left(\frac{\partial f}{\partial x}\right)(-2st)} + \cancel{\left(\frac{\partial f}{\partial y}\right)(2st)} = 0$$

4) a) $f(x,y) = x^2y + 2xy^2 + 5y^3$

Let $t \in \mathbb{R}$. Then

$$\begin{aligned} f(tx,ty) &= (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 \\ &= t^3x^2y + 2t^3xy^2 + 5t^3y^3 \\ &= t^3(x^2y + 2xy^2 + 5y^3) \\ &= t^3 f(x,y) \end{aligned}$$

$\Rightarrow f$ is homogeneous of degree 3.

b) Suppose f is homogeneous of degree n, so:

$$f(tx, ty) = t^n f(x,y) \text{ for } \underline{\text{all}}\ t$$

Using the Chain Rule, differentiate both sides with respect to t:

$$\frac{d}{dt} f(tx, ty) = \frac{d}{dt}\left(t^n f(x,y)\right)$$

$$\frac{d}{dt} f(tx,ty) = \frac{\partial f}{\partial x}(tx,ty)\,\frac{d(tx)}{dt} + \frac{\partial f}{\partial y}(tx,ty)\,\frac{d(ty)}{dt}$$

$$= \left(\frac{\partial f}{\partial x}(tx,ty)\right)\cdot x + \left(\frac{\partial f}{\partial y}(tx,ty)\right)\cdot y$$

$$\frac{d}{dt}\left(t^n f(x,y)\right) = n t^{n-1} f(x,y)$$

$$\Rightarrow \boxed{\left(\frac{\partial f}{\partial x}(tx,ty)\right)\cdot x + \left(\frac{\partial f}{\partial y}(tx,ty)\right)\cdot y = n t^{n-1} f(x,y)} \text{ holds for } \underline{\underline{\text{all}}}\ t \in \mathbb{R}$$

Let $t=1$ above and obtain:

$$\boxed{\left(\frac{\partial f}{\partial x}(x,y)\right)x + \left(\frac{\partial f}{\partial y}(x,y)\right)y = n f(x,y)} \quad \text{or} \quad \boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)}$$

$\blacksquare$

5 $f(x,y)=x^2+\sin(xy)$; find $\vec{u}$ s.t. $(D_{\vec{u}}f)_{(1,0)}=1$

$\nabla f(x,y)=\langle 2x+y\cos(xy),\ x\cos(xy)\rangle$

$\nabla f(1,0)=\langle 2,1\rangle$

$\vec{u}=\langle a,b\rangle$ <u>unit vector</u>

$$\begin{aligned}(D_{\vec{u}}f)_{(1,0)} &= \nabla f(1,0)\cdot\vec{u}\\ &= \langle 2,1\rangle\cdot\langle a,b\rangle\\ &= 2a+b\end{aligned}$$

$$\begin{cases} 2a+b=1 & \Rightarrow b=1-2a \\ a^2+b^2=1 & \Rightarrow a^2+(1-2a)^2=1 \end{cases}$$

$$a^2+\cancel{1}-4a+4a^2=\cancel{1}$$

$$5a^2-4a=0$$

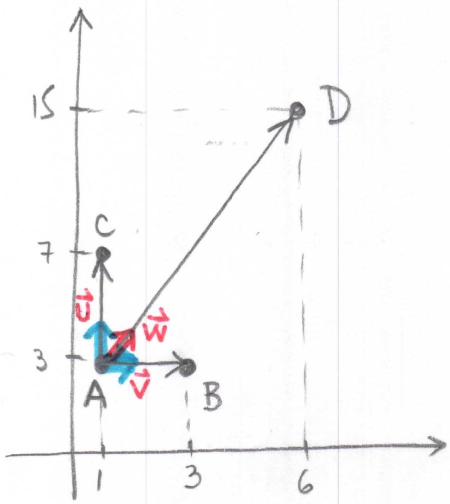
$$a(5a-4)=0 \Rightarrow a=0 \text{ or } a=4/5$$

$a=0 \Rightarrow b=1$

$a=\frac{4}{5} \Rightarrow b=1-\frac{8}{5}=-\frac{3}{5}$

$$\boxed{\vec{u}=\langle 0,1\rangle \text{ or } \vec{u}=\langle 4/5,-3/5\rangle}$$

⑥ $f(x,y)$; $A(1,3)$, $B(3,3)$, $C(1,7)$, $D(6,15)$.



Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be the directions of the vectors $\vec{AC}$, $\vec{AB}$, $\vec{AD}$, respectively.

The problem gives us that

$$(D_{\vec{u}}f)_A = 3$$

$$(D_{\vec{v}}f)_A = 26$$

and asks us to find $(D_{\vec{w}}f)_A = ?$

$$\left.\begin{array}{l}\vec{AB} = \langle 2,0\rangle \\ \|\vec{AB}\| = 2\end{array}\right\} \Rightarrow \vec{u} = \langle 1,0\rangle = \vec{i} \Rightarrow (D_{\vec{u}}f)_A = (D_{\vec{i}}f)_A = 3$$

and the directional derivative in the direction of $\vec{i}$ is actually just the partial derivative wrt. x, so this really means: $\boxed{\left(\frac{\partial f}{\partial x}\right)_A = 3}$

$$\left.\begin{array}{l}\vec{AC} = \langle 0,4\rangle \\ \|\vec{AC}\| = 4\end{array}\right\} \Rightarrow \vec{v} = \langle 0,1\rangle = \vec{j} \Rightarrow (D_{\vec{v}}f)_A = (D_{\vec{j}}f)_A = 26$$

Since the directional derivative in the $\vec{j}$ direction is just the partial derivative wrt y, this really means: $\boxed{\left(\frac{\partial f}{\partial y}\right)_A = 26}$

$$\Rightarrow (\nabla f)_A = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_A = \langle 3, 26\rangle.$$

$$\left.\begin{array}{l}\vec{AD} = \langle 5,12\rangle \\ \|\vec{AD}\| = 13\end{array}\right\} \Rightarrow \vec{w} = \left\langle \tfrac{5}{13}, \tfrac{12}{13}\right\rangle$$

$$\Rightarrow (D_{\vec{w}}f)_A = (\nabla f)_A \cdot \vec{w} = \langle 3,26\rangle \cdot \left\langle \tfrac{5}{13}, \tfrac{12}{13}\right\rangle$$

$$= \frac{15}{13} + 24 = \boxed{\frac{327}{13}}$$