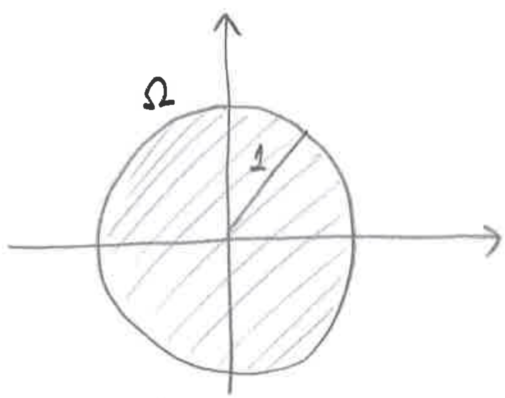
① $g(x,y) = 2x^2 + x + 2y^2 - 2$

Min/max on $\Omega = \{(x,y) : x^2+y^2 \le 1\}$

<u>Critical Points</u>: $\begin{cases} g_x = 0 \\ g_y = 0 \end{cases}$ $\begin{cases} 4x+1=0 \\ 4y=0 \end{cases}$ $\begin{cases} x = -1/4 \\ y = 0 \end{cases}$ $\boxed{(-1/4, 0)}$ belongs inside Ω

so find $g(-1/4, 0)$

$$g(-1/4,0) = \frac{2}{16} - \frac{1}{4} - 2 = -\frac{17}{8}$$



(x,y)	$g(x,y)$	
$(-1/4, 0)$	$-17/8$	← min
$(1,0)$	1	← max
$(-1,0)$	-1	

<u>Boundary</u>: To find min & max on the boundary circle $x^2+y^2=1$, we parametrize it (to get a function of one variable on a closed bounded interval):

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, \quad 0 \le \theta \le 2\pi$$

So $g(x,y)$ on the boundary becomes

$$h(\theta) = g(\cos\theta, \sin\theta)$$
$$= 2\cos^2\theta + \cos\theta + 2\sin^2\theta - 2$$
$$= 2 + \cos\theta - 2$$
$$= \cos\theta, \quad 0 \le \theta \le 2\pi$$

Critical points of $h(\theta)$ in $[0, 2\pi]$:

$h'(\theta) = -\sin\theta$

$h'(\theta) = 0, \ 0 \le \theta \le 2\pi \Rightarrow \boxed{\theta = 0, \pi, 2\pi}$

boundary points

critical point

$\theta = 0 \Rightarrow \begin{cases} x = \cos 0 \\ y = \sin 0 \end{cases} \Rightarrow \boxed{(1,0)}$ $g(1,0) = 2+1-2 = 1 = h(0)$
(or just find $h(0)$)

$\theta = \pi \Rightarrow \begin{cases} x = \cos\pi \\ y = \sin\pi \end{cases} \Rightarrow \boxed{(-1,0)}$ $g(-1,0) = 2-1-2 = -1 = h(\pi)$
(or just find $h(\pi)$)

$\theta = 2\pi \Rightarrow (1,0)$ again

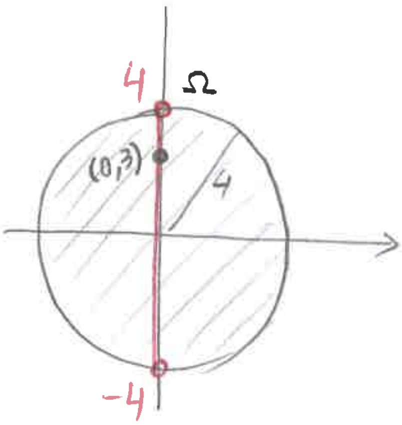
Min $= -17/8$ occurs at $(-1/4, 0)$

Max $= 1$ occurs at $(1, 0)$

(2) $f(x,y) = 2x^2 - y^2 + 6y$; $\Omega = \{(x,y) : x^2+y^2 \le 16\}$

Critical points: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\begin{cases} 4x = 0 \\ -2y+6=0 \end{cases}$ $\begin{cases} x=0 \\ y=3 \end{cases}$ $\boxed{(0,3)} \to$ inside Ω

$f(0,3) = -9+18 = 9$



Boundary: $\begin{cases} x = 4\cos\theta \\ y = 4\sin\theta \end{cases}$ $0 \le \theta \le 2\pi$

$g(\theta) = f(4\cos\theta, 4\sin\theta)$

$= 2\cdot 16\cos^2\theta - 16\sin^2\theta + 24\sin\theta$

$= 32\cos^2\theta - 16\sin^2\theta + 24\sin\theta$

(no obvious way to simplify, $g'(\theta) \overset{?}{=} 0$ looks complicated to solve for θ)

(x,y)	$f(x,y)$	
$(0,3)$	9	
$(\sqrt{15},1)$	35	max
$(-\sqrt{15},1)$	35	
$(0,-4)$	$\boxed{-40}$	$\leftarrow$ min
$(0,4)$	8	

Another way: the boundary points satisfy $x^2+y^2=16$
$f(x,y)$ contains just a $2x^2$ for the x-variable, which can be easily expressed in terms of y on the boundary: $x^2 = 16-y^2$. So we can express $f(x,y)$ as a function of y on the boundary, and $-4 \le y \le 4$ is the domain.

$g(y) = 2(16-y^2) - y^2 + 6y = 32 - 2y^2 - y^2 + 6y = -3y^2 + 6y + 32$, $-4 \le y \le 4$

$g'(y) = -6y + 6$

$g'(y) = 0 \Rightarrow y = 1$ (belongs to $[-4,4]$) ; Test g at the c.pt. 1 and endpoints:

$g(1) = -3+6+32 = 35$ occurs at $y=1 \Rightarrow x^2 = 16 - 1^2 \Rightarrow x^2 = 15 \Rightarrow x = \pm\sqrt{15}$
$(\sqrt{15}, 1)$; $(-\sqrt{15}, 1)$

$g(-4) = -48 - 24 + 32 = -40$; $y=-4 \Rightarrow x^2 = 16-16 = 0 \Rightarrow x = 0$; $(0,-4)$

$g(4) = -48+24+32 = 8$; $y = 4 \Rightarrow x^2 = 16-16=0 \Rightarrow x=0$; $(0,4)$

Absolute max: 35 occurs at $(\sqrt{15},1)$ & $(-\sqrt{15},1)$; Absolute min: -40 occurs at $(0,-4)$.

Yet another way: We are trying to find the min & max of $f(x,y) = 2x^2 - y^2 + 6y$ on the boundary, $x^2 + y^2 = 16$. This can also be done using Lagrange multipliers:

$$f(x,y) = 2x^2 - y^2 + 6y \quad \longleftarrow \text{minimize / maximize}$$

$$g(x,y) = x^2 + y^2 = 16 \quad \longleftarrow \text{constraint}$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 16 \end{cases} \qquad \begin{cases} \langle 4x, -2y+6 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 16 \end{cases}$$

$$\begin{cases} 4x = \lambda(2x) \rightsquigarrow 2x(2-\lambda) = 0 \Rightarrow x = 0 \text{ or } \lambda = 2 \\ -2y + 6 = \lambda(2y) \\ x^2 + y^2 = 16 \end{cases}$$

$$x = 0 \Rightarrow \begin{cases} -2y + 6 = 2\lambda y \\ y^2 = 16 \end{cases} \Rightarrow y = \pm 4 \Rightarrow \boxed{\begin{matrix} (0,4) \\ (0,-4) \end{matrix}}$$

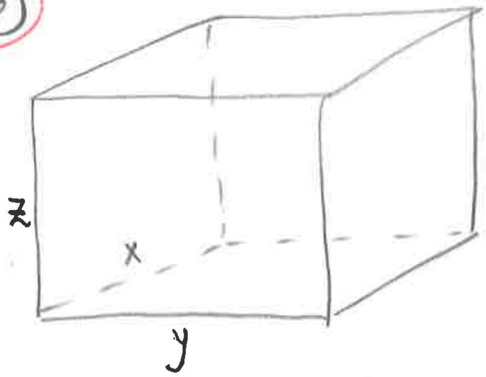
$$\lambda = 2 \Rightarrow \begin{cases} -2y + 6 = 4y \\ x^2 + y^2 = 16 \end{cases} \Rightarrow \begin{cases} y = 1 \\ x^2 + y^2 = 16 \end{cases} \Rightarrow \begin{cases} y = 1 \\ x^2 + 1 = 16 \end{cases}$$

$$\Rightarrow \begin{cases} y = 1 \\ x = \pm\sqrt{15} \end{cases}$$

$$\Rightarrow \boxed{(\sqrt{15}, 1) \,;\, (-\sqrt{15}, 1)}$$

(we get the same four points to test on the boundary: $(0, \pm 4)$ and $(\pm\sqrt{15}, 1)$).

3



Box with dimensions x, y, z

Largest volume $\Rightarrow$ maximize

$$f(x,y,z) = xyz$$

Surface area $= 64$ $\Rightarrow$ constraint:

$$2xy + 2yz + 2xz = 64$$

$$xy + yz + xz = 32$$

$$\boxed{g(x,y,z) = xy + yz + xz = 32}$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 32 \end{cases}$$

$$\begin{cases} \langle yz, xz, xy \rangle = \lambda \langle y+z, x+z, x+y \rangle \\ xy + yz + xz = 32 \end{cases}$$

$$\begin{cases} yz = \lambda(y+z) & (1) \\ xz = \lambda(x+z) & (2) \\ xy = \lambda(x+y) & (3) \\ xy + yz + xz = 32 & (4) \end{cases}$$

Multiply equation (1) by x, eqn. (2) by y, eqn (3) by z:

$(1')$: $xyz = \lambda(xy + xz)$

$(2')$: $xyz = \lambda(xy + yz)$

$(3')$: $xyz = \lambda(xz + yz)$

$\Rightarrow \lambda(xy + xz) = \lambda(xy + yz) \Rightarrow \lambda(xz - yz) = 0 \Rightarrow \lambda z(x - y) = 0$, $\lambda \neq 0$, $z \neq 0$ $\Rightarrow$ $x = y$

$\Rightarrow \lambda(xy + yz) = \lambda(xz + yz)$

$\Rightarrow \lambda(xy - xz) = 0$

$\Rightarrow \lambda x(y - z) = 0$, $\lambda \neq 0$, $x \neq 0$ $\Rightarrow$ $y = z$

$x = y = z$ $\Rightarrow$ (4) becomes: $3x^2 = 32 \Rightarrow x^2 = \frac{32}{3} \Rightarrow x = \pm\sqrt{\frac{32}{3}} = \pm\frac{4\sqrt{2}}{\sqrt{3}}$, $x > 0$ $\Rightarrow$

dimensions are $\boxed{x = y = z = \frac{4\sqrt{2}}{\sqrt{3}}}$ (cm)