Math 2401
Exam 2
Section B
Number

Name: Metric

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:

Problem	Possible	Earned
1	5	
2	5	
3	5	
4	5	
5	10	
6	10	
7	10	
Total	50	

1. (5 pts) Determine if the following limits exist and if so compute the value.

(a) (2 pts) $\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$;

(b) (3 pts) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

(a) $\frac{x-y}{x^4-y^4} = \frac{x-y}{(x^2-y^2)(x^2+y^2)} = \frac{1}{(x+y)(x^2+y^2)}$

$\Rightarrow \lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \frac{1}{4(4+4)} = \frac{1}{32}$

- 1 pt for Correct Answer
- 1 pt for Algebraic Simpl.

(b) $\boxed{\text{Limit does not exist}}$ +1

Let $y = kx^2$

- Two paths +1

$\left.\frac{x^2y}{x^4+y^2}\right|_{y=kx^2} = \frac{kx^2 \cdot x^2}{x^4(1+k^2)} = \frac{k}{1+k^2}$

So $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist since different limits on different paths +1

2. (5 pts) For the function $w(x,y) = \sin(2x-y)$ and $x(r,s) = r + \sin s$ and $y(r,s) = rs$ compute $\frac{\partial w}{\partial r}$.

$\frac{\partial w}{\partial x} = 2\cos(2x-y)$ $\qquad \frac{\partial x}{\partial r} = 1$ $\qquad 2x - y = 2r + 2\sin s - rs$

$\frac{\partial w}{\partial y} = -\cos(2x-y)$ $\qquad \frac{\partial y}{\partial r} = s$

$$\frac{\partial w}{\partial r} = \left.\frac{\partial w}{\partial x}\right|_{(x,y)} \frac{\partial x}{\partial r} + \left.\frac{\partial w}{\partial y}\right|_{(x,y)} \frac{\partial y}{\partial r} = 2\cos(2r+2\sin s - rs) - s\cos(2r+2\sin s - rs)$$

$$= (2-s)\cos(2r+2\sin s - rs)$$

- 1 pt for Correct Answer
- 1 pt for Chain Rule formula
- 1 pt $\frac{\partial w}{\partial x}$
- 1 pt $\frac{\partial w}{\partial y}$
- 1 pt $\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}$

3. (5 pts) In what direction is the derivative of the function $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ at the point $(1, 1)$ equal to zero?

$$\nabla f = \left(\frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2}, \frac{-2y(x^2+y^2) - (x^2-y^2)2y}{(x^2+y^2)^2} \right)$$

$$= \left(\frac{4xy^2}{(x^2+y^2)^2}, \frac{-4yx^2}{(x^2+y^2)^2} \right) \qquad \nabla f(1,1) = \left(\frac{4}{2^2}, \frac{-4}{2^2} \right) = (1,-1)$$

$$\underbrace{D_u f = \overbrace{\nabla f \cdot u = 0}^{+1\,pt}}_{1\,pt} \iff u \perp \nabla f \qquad u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ or } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$\boxed{\bullet +1 \text{ pt for } u}$

$\boxed{\begin{array}{ll} \bullet\ \nabla f \cdot & 1\,pt \\ \bullet\ \nabla f(1,1) & 1\,pt \end{array}}$

4. (5 pts) For the equation $x^2 + y^2 - 2xy - x + 3y - z = -4$ and the point $(2, -3, 18)$ find:

(a) the gradient;

(b) the tangent plane at the point;

(c) the normal line at the point.

$$g(x,y,z) = x^2+y^2-2xy-x+3y-z+4=0$$

(a) $\nabla g = (2x-2y-1, 2y-2x+3, -1) \longrightarrow 1\,pt$

$\nabla g(2,-3,18) = (4+6-1, -4-6+3, -1) = (9,-7,-1) \rightarrow 1\,pt$

(b) Tangent Plane $\underline{\hat{N}\cdot(x-p)=0} \longrightarrow 1\,pt$

$(9,-7,-1)\cdot((x,y,z)-(2,-3,18)) = 0$

$\iff 9x-7y-z = 18+21-18 = 21$

$\boxed{9x-7y-z=21} \longrightarrow 1\,pt$

(c) Normal line

$\ell(t) = (2,-3,18) + t(9,-7,-1) \longrightarrow 1\,pt$

5. (10 pts) Find the absolute maximum and minimum values of

$$g(x, y) = 2x^2 + x + 2y^2 - 2$$

on the set

$$\Omega = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Critical Points

+2pt $\left\{ \nabla g(x,y) = (4x+1,\ 4y) \right.$ $\qquad \nabla g = (0,0) \iff y = 0$

$4x + 1 = 0 \Rightarrow x = -\frac{1}{4}$

Critical Point $\underbrace{\left(-\frac{1}{4}, 0\right)}_{+1\,pt}$ $\qquad g\left(-\frac{1}{4}, 0\right) = 2 \cdot \frac{1}{16} - \frac{1}{4} + 0 - 2$

$= \frac{1}{8} - \frac{2}{8} - 2 = -\frac{1}{8} - 2 = \underbrace{-\frac{17}{8} = g\left(-\frac{1}{4}, 0\right)}_{+1\,pt}$

+2pt $\left\{ \text{Parameterize Boundary: } x^2 + y^2 = 1 \right.$ $\qquad x = \cos\theta \quad 0 \leq \theta \leq 2\pi$

$y = \sin\theta$

+3pts $\Bigg\{$ Restrict g to boundary : $2 + \cos\theta - 2 = g(\cos\theta, \sin\theta)$

- Extremize $\boxed{\tilde{g}(\theta) = \cos\theta \quad 0 \leq \theta \leq 2\pi}$ +1pt

$\tilde{g}'(\theta) = -\sin\theta \qquad -\sin\theta = 0 \iff \theta = 0, \pi, 2\pi$ $\Big\}$ +1pt

Critical boundary points

$\tilde{g}(0) = 1, \quad \tilde{g}(\pi) = -1, \quad \tilde{g}(2\pi) = 1 \Big\}$ +1pt

Maximum @ $(1, 0)$ of 1

Minimum @ $\left(-\frac{1}{4}, 0\right)$ of $-\frac{17}{8}$ $\Big\}$ +1pt

6. (10 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z$$

subject to the constraint that

$$x^2 + y^2 + z^2 = 25. \quad \Rightarrow g(x,y,z) = x^2+y^2+z^2-25$$

$+1\,pt\ \{\ \nabla f(x,y,z) = \frac{1}{\sqrt{3}}(1,1,1)$

$+1\,pt\ \{\ \nabla g(x,y,z) = (2x, 2y, 2z)$

$$\nabla f = \lambda \nabla g \iff \left.\begin{aligned} \frac{1}{\sqrt{3}} &= 2\lambda x \\ \frac{1}{\sqrt{3}} &= 2\lambda y \\ \frac{1}{\sqrt{3}} &= 2\lambda z \end{aligned}\right\} \underline{+1\,pt}$$

$$\Rightarrow \overbrace{x = y = z = \frac{1}{2\sqrt{3}\lambda}}^{+1\,pt} \Rightarrow \left(\frac{1}{2\sqrt{3}\lambda}, \frac{1}{2\sqrt{3}\lambda}, \frac{1}{2\sqrt{3}\lambda}\right)$$ need to satisfy constraint

$$x^2 = y^2 = z^2 = \frac{1}{3\cdot 4\lambda^2} \Rightarrow 25 = x^2+y^2+z^2 = \frac{3}{3\cdot 4\lambda^2} = \frac{1}{4\lambda^2}$$

$$\left.\Rightarrow \lambda^2 = \frac{1}{100} \Rightarrow \lambda = \pm\frac{1}{10}\ \right\} +2\,pts$$

Candidate Points: $\left.\begin{aligned}&\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)\\ &\left(-\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right)\end{aligned}\right\}$ +1 pt Each

$$f\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right) = 3\cdot\frac{5}{\sqrt{3}\sqrt{3}} = 5$$

$$f\left(-\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right) = -5$$

$$\boxed{\begin{aligned}&\text{Maximum is } 5 \text{ @ } \left(\tfrac{5}{\sqrt{3}}, \tfrac{5}{\sqrt{3}}, \tfrac{5}{\sqrt{3}}\right)\\ &\text{Minimum is } -5 \text{ @ } \left(-\tfrac{5}{\sqrt{3}}, -\tfrac{5}{\sqrt{3}}, -\tfrac{5}{\sqrt{3}}\right)\end{aligned}}$$

+1 pt each for max/min.

7. (10 pts) For the function $f(x, y) = x^3 + \frac{3}{2}y^2 + 3x^2 - 3y$:

(a) Compute the critical points;

(b) For each critical point found from part (a) determine if the function achieves a local maximum, local minimum or a saddle point by using the second derivative test.

(a) $\left.\begin{aligned}\nabla f(x,y) &= (3x^2+6x,\ 3y-3) \\ &= (3x(x+2),\ 3(y-1))\end{aligned}\right\}\ \boxed{+1\ pt}$

$\nabla f = 0 \iff 3x(x+2) = 0 \iff x=0$ or $x=-2$

$y=1 \qquad y=1$

Critical Points $(0,1), (-2,1) \} \boxed{+2\ pt}$

(b) $\frac{\partial f}{\partial x} = 3x^2 + 6x$

$\frac{\partial f}{\partial y} = 3y - 3$

$\frac{\partial^2 f}{\partial x^2} = 6x + 6$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

$\frac{\partial^2 f}{\partial y^2} = 3$

+3 pts, +1 pt Each

$$\text{Hess}(f)(x,y) = \begin{pmatrix} 6x+6 & 0 \\ 0 & 3 \end{pmatrix}$$

For point $(0,1)$

$\text{Hess}\, f(0,1) = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \xrightarrow{\text{E-values} > 0}$ local Min

+1 pt

+ 2 pts - +1 pt for local min
· +1 pt for 2nd Derivative Test

+ 2 pts

For point $(-2,1)$

$\text{Hess}(-2,1) = \begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix}$

E-Values Mixed → Saddle Point