Math 2401
Exam 1

Name: Solutions

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:____________________

Problem	Possible	Earned
1	5	
2	5	
3	10	
4	5	
5	5	
6	10	
7	10	
Total	50	

1. (5 pts) Determine the line through the point $p = (1, 2, 3)$ that is parallel to the vector $v = (-3, 0, 7)$.

$$\ell(t) = \vec{p} + t\vec{v}$$ 1 pt

$$= (1,2,3) + t(-3,0,7)$$ +1pt 1pt

$$\boxed{\ell(t) = (1-3t,\ 2,\ 3+7t)}$$ 2 pts

2. (5 pts) Compute the angle between the vectors $v_1 = (2, 1, 0)$ and $v_2 = (1, 2, -1)$. You may leave your answer un-simplifed.

$$|V_1| = \sqrt{2^2 + 1^2} = \sqrt{5} \qquad |V_2| = \sqrt{1+4+1} = \sqrt{6}$$

$$U_1 = \frac{V_1}{|V_1|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \qquad U_2 = \frac{V_2}{|V_2|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

$$\theta = \cos^{-1}(U_1 \cdot U_2) = \cos^{-1}\left(\frac{2}{\sqrt{30}} + \frac{2}{\sqrt{30}}\right)$$

$$= \boxed{\cos^{-1}\left(\frac{4}{\sqrt{30}}\right) = \theta}$$

$U_1 \cdot U_1 = 2$ pts $\qquad U_2 = 1$ pt

$U_1 = 1$ pt $\qquad \theta = \cos^{-1}(\dots)$ 1 pt

3. (10 pts) Let $v_1 = (2, 4, 5)$, $v_2 = (1, 5, 7)$ and $v_3 = (-1, 6, 8)$.

(a) (3 pts) Compute $u_1 = v_1 - v_3$ and $u_2 = v_2 - v_3$;

(b) (3 pts) Compute $u_1 \times u_2$;

(c) (4 pts) Determine the plane through the points v_1, v_2, and v_3.

(a) $U_1 = V_1 - V_2 = (2,4,5) - (-1,6,8)$

$= \boxed{(3,-2,-3) = U_1}$

2 pts for correct answer
1 pt for set up

$U_2 = V_2 - V_3 = (1,5,7) - (-1,6,8)$

$= \boxed{(2,-1,-1) = U_2}$

(b) $U_1 \times U_2 = \underbrace{\det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & -3 \\ 2 & -1 & -1 \end{pmatrix}}_{\boxed{2\ pt}} = \hat{\imath}(2-3) - \hat{\jmath}(-3+6) + \hat{k}(-3+4)$

$= (-1,-3,1)$ $\boxed{1\ pts}$

(c) $\hat{N} = (-1,-3,+1)$

$\vec{p} = V_3 = (-1,6,8)$

2 pts

$\boxed{\hat{N} \cdot (\hat{x} - \hat{p}) = 0 \quad \text{Equation of plane}}$

$\hat{N} \cdot \hat{x} = -x - 3y + z \longrightarrow 1\ pt$

$\hat{N} \cdot p = (-1,-3,1) \cdot (-1,6,8)$

$= 1 - 18 + 8 = -9 \longrightarrow 1\ pt$

$\boxed{-x - 3y + z = -9}$ $\boxed{1\ pt}$

4. (5 pts) Evaluate the integral:

$$\int_0^1 \left[te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right] dt.$$

- 1 pt each component
- 2 pts for correct answer

- $\frac{1}{2}\int_0^1 2te^{t^2}\,dt = \frac{1}{2}\int_0^1 e^u\,du = \frac{e^u}{2}\Big|_0^1 = \frac{1}{2}(e-1)$
- $\int_0^1 e^{-t}\,dt = -e^{-t}\Big|_0^1 = -(e^{-1}-1) = 1-e^{-1}$
- $\int_0^1 dt = 1$

$$\int_0^1 \left[te^{t^2}\hat{i} + e^{-t}\hat{j} + \hat{k}\right] dt = \left(\frac{1}{2}(e-1),\ 1-e^{-1},\ 1\right)$$

5. (5 pts) If $\mathbf{r}(t) = (e^{-t}, 2\cos(3t), 2\sin(3t))$ compute the velocity and acceleration vectors.

$r(t) = (e^{-t}, 2\cos 3t, 2\sin 3t)$

$r'(t) = (-e^{-t}, -6\sin 3t, 6\cos 3t)$ 2 pts

$r''(t) = (e^{-t}, -18\cos 3t, -18\sin 3t)$ 3 pts

6. (10 pts) Find the length of the curve

$$\mathbf{r}(t) = \left(t\cos t, t\sin t, \frac{2\sqrt{2}}{3} t^{\frac{3}{2}} \right)$$

from $(0,0,0)$ to $\left(-\pi, 0, \frac{2\sqrt{2}}{3}\pi^{\frac{3}{2}}\right)$.

$$r(0) = (0\cdot 1,\ 0\cdot 0,\ 0) = (0,0,0)$$

$$r(\pi) = \left(\pi\cos\pi,\ \pi\sin\pi,\ \frac{2\sqrt{2}}{3}\pi^{3/2}\right) = \left(-\pi,\ 0,\ \frac{2\sqrt{2}}{3}\pi^{3/2}\right)$$

$$r'(t) = \left(\cos t - t\sin t,\ \sin t + t\cos t,\ \sqrt{2}\, t^{\frac{1}{2}}\right)$$

$$|r'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t}$$

$$= \sqrt{2t + \cos^2 t \cancel{- 2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t \cancel{+ 2t\cos t\sin t}}$$

$$= \sqrt{2t + 1 + t^2} = (t+1)$$

$$\text{Length} = \int_0^\pi |r'(t)|\, dt = \int_0^\pi (t+1)\, dt = \frac{t^2}{2} + t \Big|_0^\pi$$

$$= \frac{\pi^2}{2} + \pi$$

- Answer : 1 pt
- Arc-length Formula : 2 pts
- $r(0) \Rightarrow t = 0$, $r(\pi) = t = \pi$ $\}$ 1 pt each
- $r'(t)$ $\}$ ~~3~~ 3 pts 1 per component
- $|r'(t)| = 2$ pts

7. (10 pts) Find the unit tangent $\mathbf{T}(t)$ and the principal normal $\mathbf{N}(t)$ of the function

$$\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, 2\right).$$

$$r'(t) = \left(e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 0\right)$$

$$|r'(t)| = \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\; e^t$$

$$= e^t \sqrt{\cos^2 t + \sin^2 t - \cancel{2\cos t \sin t} + \sin^2 t + \cos^2 t + \cancel{2\cos t \sin t}}$$

$$= \sqrt{2}\, e^t$$

$$\Rightarrow \boxed{T(t) = \frac{1}{\sqrt{2}}\left(\cos t - \sin t, \sin t + \cos t, 0\right)}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{\sqrt{2}}\left(-\sin t - \cos t, \cos t - \sin t, 0\right)$$

$$T'(t) = \frac{1}{\sqrt{2}}\left(-\sin t - \cos t, \cos t - \sin t, 0\right)$$

$$|T'(t)| = 1$$

- T +1 formula
- N +1 formula
- Derivate +2 $< T', r'$
- $|T'|, |r'|$ +1 pt each
- Final Answer +2