1. (5 pts) Sketch the region of integration and write an equivalent integral with the order of integration reversed:

$$\int_0^{\frac{3}{2}} \left(\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} f(x,y)\,dx \right) dy$$

Sketch:

$0 \le y \le \frac{3}{2}$

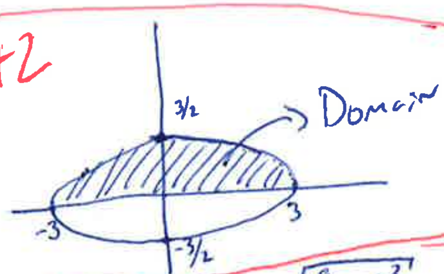
$-\sqrt{9-4y^2} \le x \le \sqrt{9-4y^2}$

$x^2+4y^2=9$

$\Rightarrow 9-x^2=4y^2$

$\Rightarrow \frac{9}{4}-\frac{x^2}{4}=y^2$

+2



+3

$$\boxed{\int_{-3}^{3} \left(\int_0^{\sqrt{\frac{9}{4}-\frac{x^2}{4}}} f(x,y)\,dy \right) dx}$$

2. (5 pts) Evaluate the integral:

$$\pi = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy \right) dx$$

$\Omega = \{(x,y) : |x|<1 \quad |y|<\sqrt{1-x^2}\} = \{(x,y) : x^2+y^2 \le 1\}$

$= \{(r,\theta) : 0 \le r \le 1 \quad 0 \le \theta \le 2\pi\}$ +2

$$\int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy \right) dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{2r}{(1+r^2)^2}\,dr\,d\theta$$ +2

$u = 1+r^2 \quad du = 2r\,dr$

$u(0)=1$

$u(1)=2$

$$= 2\pi \int_0^1 \frac{2r}{(1+r^2)^2} dr$$

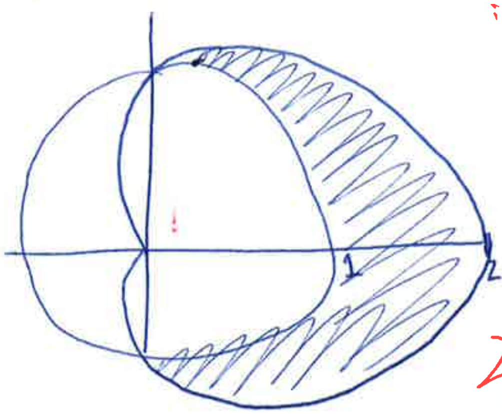
+1

$$= 2\pi \cdot \int_1^2 \frac{du}{u^2} = 2\pi \cdot -\frac{1}{u}\Big|_1^2 = 2\pi\left(-\frac{1}{2}+1\right) = \boxed{\pi}$$

3. (10 pts) Let D be the solid whose base is the region in the xy-plane that lies inside the cardiod $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$. You must sketch the base of the domain, and will receive +1 bonus for a convincing sketch of the solid D.

Sketch of Base

$$\text{Base} = \{(r,\theta) : 1 \le r \le 1 + \cos\theta, \; -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\}$$



3

$$1 + \cos\theta = 1 \iff \theta = \tfrac{\pi}{2}, -\tfrac{\pi}{2}$$

$$\boxed{\text{Volume}(D) = 8 + \pi}$$

$$\text{Volume}(D) = \iiint_D dV(x,y,z) = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \left(\int_{r=1}^{1+\cos\theta} \left(\int_{z=0}^{4} dz \right) r\,dr \right) d\theta$$

3

1

$$= 4 \int_{-\frac{\pi}{2}}^{\pi/2} \left(\int_{1}^{1+\cos\theta} r\,dr \right) d\theta = \frac{4}{2} \int_{-\frac{\pi}{2}}^{\pi/2} \left((1+\cos\theta)^2 - 1 \right) d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\frac{1+\cos 2\theta}{2} = \cos^2\theta$

$$= 2 \cdot \int_{-\pi/2}^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$

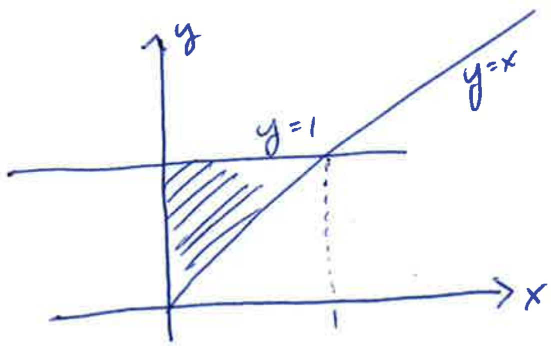
3

$$= 4 \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta + \frac{2}{2} \int_{-\frac{\pi}{2}}^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= 4 \cdot \sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \theta \Big|_{-\frac{\pi}{2}}^{\pi/2} + \frac{1}{2}\sin 2\theta \Big|_{-\frac{\pi}{2}}^{\pi/2} = 8 + \pi$$

4. (10 pts) Compute the volume of the solid D that is bounded above by the plane $z = 2-x$, and whose base in the xy-plane is bounded by the y-axis and the lines $y = x$ and $y = 1$. You must sketch the base of the domain, and will receive +1 bonus for a convincing sketch of the solid D.

Sketch of base



1 point final answer
2 points sketch
3 points bounds
4 points integration
{1 point correct order
1 point each integral
(if the integral is not simplified by a previous mistake)

$$\text{Base} = \{(x,y) : 0 \le x \le 1 \ , \ x \le y \le 1\}$$

$$\text{Volume}(D) = \int_{x=0}^{1} \left(\int_{y=x}^{1} \left(\int_{z=0}^{2-x} dz \right) dy \right) dx$$

$$= \int_{x=0}^{1} \left(\int_{y=x}^{1} (2-x)\, dy \right) dx$$

$$= \int_{x=0}^{1} (1-x)(2-x)\, dx = \int_0^1 (2-2x-x+x^2)\, dx$$

$$= 2x - \frac{3}{2}x^2 + \frac{x^3}{3} \Big|_0^1 = 2 - \frac{3}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

5. (10 pts) Let T be the region with $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, and $0 \le z \le \sqrt{1-x^2-y^2}$.

(a) Sketch T;

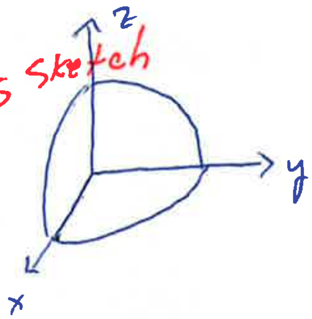
(b) Evaluate

$$\frac{\pi}{2} = \iiint_T \frac{1}{x^2+y^2+z^2} dV(x,y,z).$$

(a) $T = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le \sqrt{1-x^2-y^2}\}$

$\Rightarrow \rho^2 \le 1,\quad 0 \le \theta \le \frac{\pi}{2},\quad 0 \le \phi \le \frac{\pi}{2}$

2pts sketch



3pts boundaries

2pts correct setup in spherical

3pts integrating

$$\iiint_T \frac{1}{x^2+y^2+z^2} dV(x,y,z) = \int_{\rho=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \frac{1}{\rho^2}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^1 d\rho\right)\left(\int_{\theta=0}^{\pi/2} d\theta\right)\left(\int_{\phi=0}^{\pi/2} \sin\phi\, d\phi\right)$$

$$= 1 \cdot \frac{\pi}{2} \cdot -\cos\phi \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(0 - (-1)\right) = \frac{\pi}{2}$$

6. (10 pts) Let Ω be the region in the xy plane that lies in the first quadrant, is bounded by the hyperbola $xy = 1$, $xy = 9$ and the lines $y = x$ and $y = 4x$.

(a) Sketch the region Ω;

(b) Find a domain Γ in the uv-plane so that the transformation $x(u,v) = \frac{u}{v}$ and $y(u,v) = uv$ with $u > 0$ and $v > 0$ maps Ω to Γ.

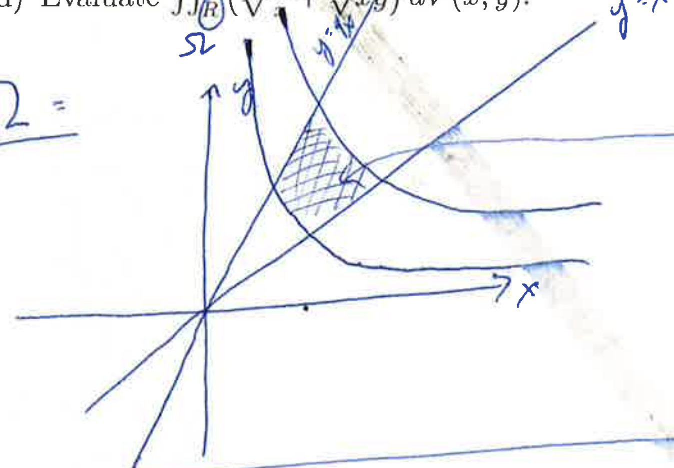
(c) Find the Jacobian of the above transformation;

(d) Evaluate $\iint_{R} \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dV(x,y)$.

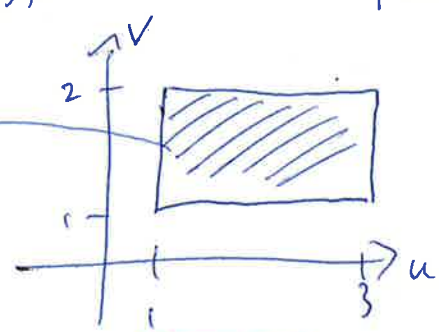
Ω

(a) $\Omega =$

+2



(b) $\Gamma = \{(u,v) : 1 \le u \le 3,\ 1 \le v \le 2\}$



$yx = \frac{u}{v} \cdot uv = u^2 \qquad \frac{y}{x} = \frac{uv}{\frac{u}{v}} = v^2$

$u^2 = xy = 1$ or 9, $1 \le u \le 3$

$v^2 = \frac{y}{x} \Rightarrow 1 \le v \le 2$

+2

(c) Jacobian

$$J(u,v) = \det \begin{pmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{pmatrix}$$

+2

$$= \det \begin{pmatrix} \frac{1}{v} & v \\ -\frac{u}{v^2} & u \end{pmatrix} = \frac{u}{v} + \frac{vu}{v^2} = \frac{2u}{v}$$

$$\text{(d)} \iint_{\Omega} \sqrt{\frac{y}{x}} + \sqrt{xy}\, dV(x,y) = \iint_{\Gamma} (u+v) \frac{2u}{v}\, dV(u,v)$$

+1 +1

$$= \int_{u=1}^{3} \int_{v=1}^{2} \left(\frac{2u^2}{v} + 2u\right) dv\, du = \left(\int_{u=1}^{3} 2u^2 du\right)\left(\int_{v=1}^{2} \frac{dv}{v}\right) + \int_{v=1}^{2} dv \int_{u=1}^{3} 2u\, du = 8 + \frac{52}{3} \ln 2$$

+1 +1