Color-position symmetry in interacting particle systems

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Plan

- (Multi-species) interacting particle systems as random walks on Hecke algebras. Color-position symmetry as involution from Hecke algebra.
- Asymptotic application: Second class particle in ASEP
- Asymptotic application: Mixing time of ASEP on a segment

Hecke algebra

$W = S_n$, $s_i = (i, i+1)$.

$L(w) :=$ number of inversions in $w \in W$.

Hecke algebra: $\{T_w\}_{w \in W}$ — linear basis

$$\begin{cases} T_s T_w = T_{sw}, & \text{if } L(sw) = L(w) + 1 \\ T_s T_w = (1-q) T_w + q T_{sw}, & \text{if } L(sw) = L(w) - 1. \end{cases}$$

The linear map $I : \mathcal{H} \to \mathcal{H}$

$$I : \sum_w a_w T_w \to \sum_w a_w T_{w^{-1}}$$

satisfies

$$I(h_r h_{r-1} \dots h_2 h_1) = I(h_1) I(h_2) \dots I(h_r), \qquad h_i \in \mathcal{H}.$$

Random walk on Hecke algebra

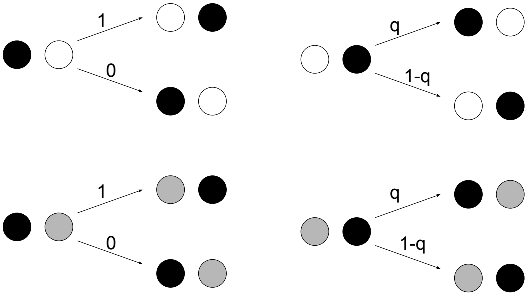
Generators $\{G_1, \ldots, G_k\}$, each of these generators has an independent exponential clock. When the clock s rings, we multiply G_s to the current position of the random walk $P \in \mathcal{H}$ — our new position is $G_s P$. This is a *random walk on Hecke algebra*.

An element of Hecke algebra

$$h := \sum_w \kappa_w T_w, \qquad \kappa_w \geq 0, \;\; \sum_w \kappa_w = 1,$$

can be interpreted as a random element of W. Random walk on Hecke algebra generates the random walk on W.

Multi-species ASEP



We consider particles of various types (=classes, colors, species).

Set of types is linearly ordered, and a particle of a smaller type interacts with a particle of a larger type as a particle with a hole.

Particular case: configurations are given by permutations $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$, where $\pi(j)$ is encoding the type of a particle standing at j.

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP: Vocabulary

- Random multi-species configuration — element of Hecke algebra
- Update —- multiplication by T_s
- ASEP evolution — element of S_n generated by random walk on Hecke algebra
- Projection to fewer colors — projection to cosets of parabolic subgroups
- Class-position symmetry — involution I swaps w and w^{-1}.

Other Coxeter groups generate ASEP with a source (hyperoctahedral group), ASEP on a ring (affine Weyl group $\tilde{A}_n$).

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP.

- Multi-species ASEP is generated by Hecke algebra: Alcaraz-Rittenberg'93, Alcaraz-Droz-Henkel-Rittenberg'93, ..., Lam'11 , Cantini-de Gier-Wheeler'15, ...
- Color-position symmetry and applications for asymptotic analysis: Angel-Holroyd-Romik'08 (TASEP, $q = 0$), Amir-Angel-Valko'08 (ASEP), Borodin-Bufetov'19 (inhomogeneous stochastic six vertex model).
  Explanation through Hecke algebra: Bufetov'20, Galashin'20; a closely related proof Kuan'20.

What happens if we consider other generators of the random walk on Hecke algebra ?

Mallows measure on S_n

S_n — symmetric group, $L(w)$ — number of inversions in w, and $0 \leq q < 1$.

$$\mathrm{Prob}\,(w) = q^{n(n-1)/2 - L(w)} Z.$$

For $q = 0$ this measure is concentrated on one word (longest element), for general q it is "not far" from it for large n.

If we run multi-species ASEP on a finite interval of length n for a long time, it converges to this measure.

Mallows'53

$n \to \infty$: Gnedin-Olshanski'09, Gnedin-Olshanski'11

Other sets of generators also lead to interesting particle systems.

$[a;b] := \{j \in \mathbb{Z} : a \leq j \leq b\}$ the interval between a and b. $S_{a;b} \subset S_n$ permutes the elements from $[a;b]$ only.

Mallows element

$$\mathcal{M}_{a;b} := \sum_{w \in S_{a;b}} Zq^{(b-a+1)(b-a)/2 - L(w)} T_w, \qquad \mathcal{M}_{a;b} \in \mathcal{H}(S_n),$$

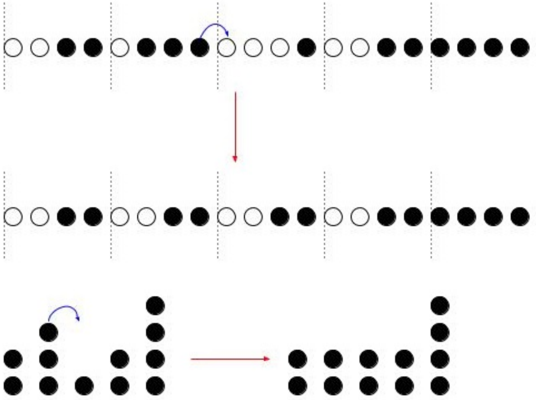
where $L(w)$ is the number of inversions in w. The main property of the element $\mathcal{M}_{a;b}$ is

$$T_w \mathcal{M}_{a;b} = \mathcal{M}_{a;b} T_w = \mathcal{M}_{a;b}, \qquad \text{for any } w \in S_{a;b}.$$

Bufetov'20: Let $n = NM$, with $M, N \in \mathbb{Z}_{>0}$, and consider the following set of generators of a random walk on the Hecke algebra :

$$\left\{\mathcal{M}_{(x-1)M+1;xM}\mathcal{M}_{xM+1;(x+1)M}\, T_{(xM,xM+1)}\mathcal{M}_{(x-1)M+1;xM}\mathcal{M}_{xM+1;(x+1)M}\right\}_{x=1}^{N-1}.$$

This dynamics generates a multi-species $ASEP(q, M)$.
$q = 0$: M-exclusion TASEP.



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This dynamics generates a multi-species $ASEP(q, M)$.

- Construction is related to the notion of fusion: Kulish-Reshetikhin-Sklyanin'81, Corwin-Petrov'15.
- Single species version of ASEP(q,M) was introduced by Carinci-Giardina-Redig-Sasamoto'15
- Multi-species version of ASEP(q,M) was introduced by Kuan'16
- $M \to \infty$: q-TAZRP (single species version introduced by Sasamoto-Wadati'98).

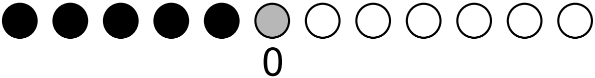
Instead of just $T_{(xM,xM+1)}$ we can have arbitrary interaction between two blocks. This leads to a variety of processes and possible interactions, and one obtains multi-species versions of all these processes.

How does the color-position symmetry help in asymptotic applications ?

Assume that we want to analyze ASEP which starts from some initial configuration. Lifting it into the multi-species ASEP (in some way), let us say that the initial configuration is given by permutation w. Then we need to study $W(t)T_w$, where $W(t)$ is a random walk which started from identity.

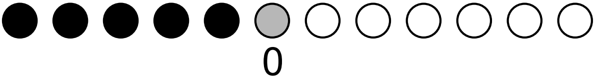
Crucial idea: One can study $\iota\,(T_{w^{-1}}W(t))$ instead — it has exactly the same distribution ! The benefit is that the continuous time process starts from identity (which leads to step initial condition under projection to fewer types). One needs to analyze the multiplication by $T_{w^{-1}}$ afterwards though....

For general w this is arguably very hard. However, for certain special choices this is quite accessible !



Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

Asymptotics of $S_1(t)$?

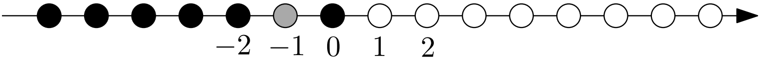


Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

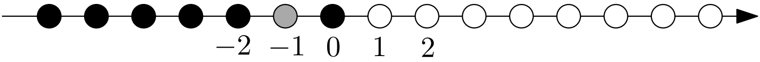
$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) = \frac{1}{2}\left(1 + \frac{x}{1-q}\right).$$

Uniform distribution on $[-(1-q); (1-q)]$.

P.A. Ferrari-Kipnis'95, P.A. Ferrari-Goncalves-Martin'08.



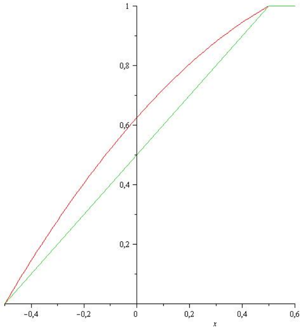
The asymptotic distribution of the second class particle ?

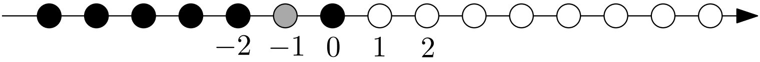


(Borodin-Bufetov'19) The asymptotic distribution of the second class particle

$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) + (1-q)d(-x)(1-d(-x)).$$

Note the nontrivial dependence on q.





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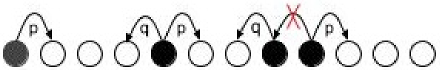
Note the nontrivial dependence on q.

- $q = 0$: TASEP, Cator-Pimentel'13: for general initial conditions.
- for a class of initial configurations and general q: Borodin-Bufetov'19
- Bufetov'20: Similar results for a second class particle for half-line ASEP with a source, and a second class particle in q-TAZRP.

ASEP on a finite interval



There are k_N particles on $\tilde{\mathbb{Z}}_N = \{1, 2, \ldots, N\}$ which evolve in time. There are two Poisson processes of rates p and $q < p$ associated with each particle, $p + q = 1$.

Each particle jumps one step to the right with rate p, and jumps one step to the left with rate q, if the neighboring positions are vacant. If the position is occupied by another particle, the jump does not happen.

All Poisson processes are independent.

Cutoff

Ergodic Markov chain with finitely many states. S— state space, ξ — initial configuration, Q_t^ξ — the distribution of the Markov chain started from ξ at time t.

There is a unique stationary distribution π. We measure the *total variance distance*:

$$||Q_t^\xi - \pi||_{TV} := \frac{1}{2} \sum_{w \in S} |Q_t^\xi(w) - \pi(w)| = \max_{A \subset S} |Q_t^\xi(A) - \pi(A)|.$$

$$d(t) := \max_{\xi \in S} ||Q_t^\xi - \pi||_{TV}$$

Mixing time:

$$T^{mix}(\varepsilon) := \inf\{t : d(t) \leq \varepsilon\}.$$

Cutoff

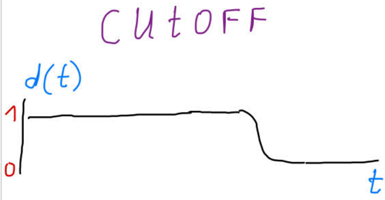
A *sequence* of Markov chains depending on N.

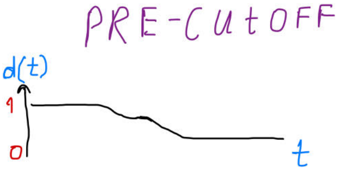
Cutoff: for any $\varepsilon > 0$:

$$\lim_{N\to\infty} \frac{T_N^{mix}(\varepsilon) - T_N^{mix}(1-\varepsilon)}{T_N^{mix}(1/4)} = 0.$$

Pre-cutoff:

$$\sup_{\varepsilon} \limsup_{N\to\infty} \frac{T_N^{mix}(\varepsilon) - T_N^{mix}(1-\varepsilon)}{T_N^{mix}(1/4)} < \infty.$$





Cutoff profile

A sequence of Markov chains exhibits a cutoff at time $f(N)$ with window of order $g(N)$ if

$$\lim_{c\to+\infty} \limsup_{N\to\infty} d_N(f(N)+cg(N)) = 0,$$

$$\lim_{c\to-\infty} \liminf_{N\to\infty} d_N(f(N)+cg(N)) = 1,$$

(for $g(N) << f(N)$).

This cutoff has profile $\mathcal{F}(c)$ if

$$\lim_{N\to\infty} d_N(f(N)+cg(N)) = \mathcal{F}(c).$$

ASEP on a finite interval

There is a unique stationary measure for ASEP on a finite interval.

Previous results

- Diaconis-Ram'00: a discrete time ASEP (systematic scan Methropolis algorithm; colored vertex model) exhibits a pre-cutoff. Method: representations of Hecke algebra.
- Benjamini-Berger-Hoffman-Mossel'02: continuous time ASEP (as defined above) exhibits a pre-cutoff. Method: link with ASEP on an infinite lattice. Hydrodynamics.
- Labbe-Lacoin'16: continuous time ASEP exhibits cutoff. Method: link with ASEP on an infinite lattice. Hydrodynamics.

Cutoff profile for ASEP

Theorem (Bufetov-Nejjar'20)

For ASEP on an interval of length N with k_N particles, assume that $k_N/N \to \alpha \in (0;1)$, as $N \to \infty$. We have

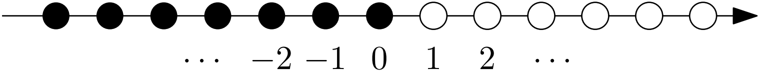
$$\lim_{N\to\infty} d_N \left(\frac{N\left(1+2\sqrt{\alpha(1-\alpha)}\right)+cN^{1/3}}{p-q} \right) = 1 - F_{GUE}\left(cf(\alpha)\right),$$

where

$$f(\alpha) := \frac{(\alpha(1-\alpha))^{1/6}}{\left(\sqrt{\alpha}+\sqrt{1-\alpha}\right)^{4/3}}.$$

and F_{GUE} is a distribution function of the (GUE) Tracy-Widom distribution.

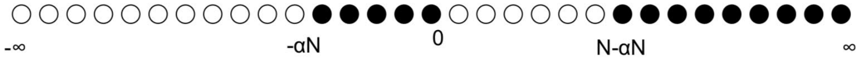
Step initial conditions



$$t(N,c) := \frac{N\left(1+2\sqrt{\alpha(1-\alpha)}\right)+cN^{1/3}}{p-q}.$$

Theorem (Tracy-Widom'08)

$$\lim_{N\to\infty} \mathrm{Prob}\left(x_{k_N}^{step}\left(t(N,c)\right) \leq N - 2k_N - o\left(N^{1/3}\right)\right) = 1 - F_{GUE}(cf(\alpha)).$$



Proposition (Bufetov-Nejjar'20)

At time $t(N, c)$

$$\mathrm{Prob}\Big(\textit{leftmost particle is at position} \geq N - 2\alpha N - N^{1/10},\ \textit{rightmost hole is at position} \leq N - 2\alpha N + N^{1/10}\Big) = F_{GUE}(cf(\alpha)).$$

How to analyze such ASEP ? It is not clear how to write observables / make comparison with step initial condition / etc.

Fortunately, the technique related to the multi-species ASEP and random walks on Hecke algebra help here.