Algebraic Duality Methods in Probability

Self-duality for particle systems via q-orthogonal polynomials

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Outline

Stochastic duality for Markov processes

Definition

Let $(x_t)_{t\geq0}$ and $(n_t)_{t\geq0}$ Markov process on Ω and Ω^{dual} with generators L and L^{dual}, respectively. x_t is **dual** to n_t with duality function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ if $\forall t \geq 0$,

$$\mathbb{E}_x(D(x_t, n)) = \mathbb{E}_n(D(x, n_t)) \qquad \forall (x, n) \in \Omega \times \Omega^{dual} .$$

At the level of the Markov generators

$$LD(\cdot, n)(x) = L^{dual} D(x, \cdot)(n)$$

Duality of processes $\Rightarrow$ duality of their generators.

Self-duality

If the two processes show the same dynamics, $L^{dual} = L$ and $\Omega = \Omega^{dual}$ we say that they are **self-dual**.
If the state space is countable, the self-duality relation can be written in matrix notation

$$LD = DL^t$$

Duality functions are not unique, but we say that two duality functions are equivalent if they coincide up to a factor that is constant under the dynamics of the process.

An algebraic approach to duality

Algebraic recipe says that, whenever there is an algebraic description of the Markov generators, then:

- (Self-)duality can be seen as a change of representation of a Lie algebra: the algebra generators are building blocks for the Markov generators and the intertwining functions between two equivalent representations yields the duality function.[1]
- Self-duality is related to the reversibility of the process and the existence of an algebra element, called symmetry, that commutes with the generator of the process.[2,3]
- Orthogonal self-duality can also arise from the scalar product method, which also helps identifying the symmetry associated.[4]

[1] Giardinà, Kurchan, Redig (2007) [2] Sandow, Schütz (1994) [3] Giardinà, Kurchan, Redig, Vafayi (2009) [4] Carinci, F., Giardinà, Groenevelt, Redig (2019)

New self-duality functions from symmetries

The idea is to construct non-trivial self-duality functions starting from the "cheap" self-duality function.

1) The cheap self-duality is always available from the reversibility of the process: $d(x, n) = \dfrac{\delta_{x,n}}{\mu(x)}$.

2) Let d be a self-duality function of the generator L and let S be a symmetry of L, then $D = Sd$ is again a self-duality function for L.

How to find S?

3i) One symmetry, S^{tr}, is suggested by the algebraic description of the Markov generator. This symmetry produces "triangular" self-duality function.

3ii) The symmetry associated to the orthogonal self-duality, S^{or}, can be found identifying the orthogonal self-duality function as scalar product of two triangular ones. In this case, the symmetry associated to the orthogonal self-duality is unitary in $L^2(\mu)$, namely
$S^{or}\,(S^{or})^* = (S^{or})^*\,S^{or} = I$

The models

The process generator is $L = \sum_{x=1}^{N-1} L_{i,i+1}$

$$L_{x,x+1}f(x) = q^{\sigma(2\theta-1)}\{x_i\}_{q^2}\{\theta+\sigma x_{i+1}\}_{q^{-2\sigma}}[f(x^{i,i+1}) - f(x)] + q^{-\sigma(2\theta-1)}\{x_{i+1}\}_{q^{-2}}\{\theta+\sigma x_i\}_{q^{2\sigma}}[f(x^{i+1,i}) - f(x)]$$

$$\{a\}_q := \frac{1-q^a}{1-q} \qquad \sigma = \begin{cases} -1 & \text{exclusion, } \theta \in \mathbb{N} \\ +1 & \text{inclusion, } \theta \in \mathbb{R}^+ \end{cases}$$

- As $q \to 1$ we find the symmetric IPS
- For $\theta = 1$ and $\sigma = -1$ we find the ASEP[5] which jump to the right with rate q^{-1} and to the left with rate q

[5] Schütz (1997)

Construction of the Markov generators [6,7]

Consider the quantum algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ with generators A^+, A^-, A^0 satisfying the commutation relations

$$\begin{cases} [A^0, A^\pm] = \pm A^\pm \\ [A^+, A^-] = -\sigma [2A^0]_q \end{cases} \qquad [A]_q := \frac{q^A - q^{-A}}{q - q^{-1}}$$

with $(A^+)^* = -\sigma A^-$ and $(A^0)^* = A^0$ and $C = A^+A^- + \left[A^0\right]_q \left[A^0 - 1\right]_q$.
Then the Markov generators can be found performing a positive ground state transformation via the diagonal matrix $G(x,n) = \delta_{x,n}\sqrt{\mu_\alpha(x)}$ of the quantum Hamiltonian defined in terms of the Casimir element.

$$H := \sum_{i=1}^{N-1} \left\{ \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes \Delta(C) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(N-i-1)\text{ times}} \right\}$$

[6,7] Carinci, Giardinà, Redig, Sasamoto (2015), (2016)

Construction of the Markov generators

The co-product is an algebra homomorphism and on the generators is given by

$$\Delta(A^0) = A^0 \otimes 1 + 1 \otimes A^0 \qquad \Delta(A^\pm) = A^\pm \otimes q^{-A^0} + q^{A^0} \otimes A^\pm$$

The following two irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$

$$\begin{cases} A^+|n\rangle &=& \sqrt{[\theta+\sigma n]_q[n+1]_q}\,|n+1\rangle \\ A^-|n\rangle &=& -\sigma\sqrt{[n]_q[\theta+\sigma(n-1)]_q}\,|n-1\rangle \\ A^0|n\rangle &=& (n+\sigma\theta/2)\,|n\rangle \end{cases}$$

In the symmetric case: $H = \Delta(C)$ and $[S^{tr}, H] = 0$ choose $S^{tr} = e^{\Delta(A^+)}$
In the asymmetric case: $H = \Delta(C)$ then $L = G^{-1}HG$ and $[S^{tr}, H] = 0$ then $G^{-1}S^{tr}G$ is a symmetry for L, where $S^{tr} = e_{q^2}\left(\sqrt{\alpha}(1-q^2)\Delta^{N-1}(q^{A^0}A^+)\right)$.
$e_{q^2}(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q^2)_n}$

Reversible measure

The reversible measure is a non-homogeneous product labeled by $\alpha \in \mathbb{R} \setminus \{0\}$

$$\mu_{\alpha,\sigma}(x) = \prod_{i=1}^{N-1} \mu_i(x_i) \quad \text{where} \quad \mu_i(x_i) = \Psi_\sigma(\theta, x_i)\, \alpha^{x_i}\, q^{2\sigma\theta i x_i}$$

for the function $\Psi_{q,\sigma}(\theta, m) := \begin{cases} \binom{\theta}{m}_q & \text{for } \sigma = -1 \\ \binom{m+\theta-1}{m}_q & \text{for } \sigma = +1 \end{cases}$

Triangular dualities

Let

$$N_i^+(x) := \sum_{m=i}^N x_m \qquad N_i^-(x) := \sum_{m=1}^i x_m \qquad N(x) := \sum_{m=1}^N x_m$$

It is known[6,7] that the following functions are the self-dualities for ASEP $(\sigma = -1)$ and ASIP $(\sigma = 1)$ associated to the symmetries S^{tr} and $\widehat{S^{tr}}$:

$$D_\lambda^{tr}(x,n) = \prod_{i=1}^N \frac{\binom{x_i}{n_i}_q}{\Psi_{q,\sigma}(\theta, n_i)} q^{x_i\left(2N_{i-1}^-(n)+n_i\right)-2\sigma\theta i n_i} \lambda^{n_i} 1_{\{n_i \le x_1\}}$$

$$\widehat{D}_\lambda^{\mathrm{tr}}(x,n) = q^{-2N(n)N(x)} D_\lambda^{tr}(x,n)$$

They differ up to a conservative quantity. Give information on the $q-$exponential moment of the integrated current in $[0,t]$: $J_i(t) = N_i^-\left(x\left(t\right)\right) - N_i^-(x)$

[6,7] Carinci, Giardinà, Redig, Sasamoto (2015), (2016)

Hypergeometric functions

Recalling that

$$ {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; q, z\right) := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k} $$

$(a;q)_k = \prod_{i=0}^{k-1}(1 - aq^i)$ are the q-shifted factorial.
For $a = q^{-n}$, $n \in \mathbb{N}$ is a polynomial of degree n in b.
${}_2\varphi_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} ; q, z\right) \longrightarrow {}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} ; z\right)$ as $q \to 1$.

Then

$$ K_n(q^{-x}; p, c; q) := {}_2\varphi_1\left(\begin{matrix} q^{-x}, q^{-n} \\ q^{-c} \end{matrix} ; q, pq^{n+1}\right) $$ the $q-$Krawtchouk polynomials

$$ M_n(q^{-x}; b, c; q) := {}_2\varphi_1\left(\begin{matrix} q^{-x}, q^{-n} \\ bq \end{matrix} ; q, -\frac{q^{n+1}}{c}\right) $$ the $q-$Meixner polynomials

Statement of the theorem

- The ASEP(q,θ) is self-dual with self-duality functions: $D_\alpha(x,n) := \prod_{i=1}^N K_{n_i}(q^{-2x_i}; p_{i,\alpha}(x,n), \theta; q^2)$
- The ASIP(q,θ) is self-dual with self-duality functions: $D_\alpha(x,n) := \prod_{i=1}^N M_{n_i}(q^{-2x_i}; q^{2(\theta-1)}, c_{i,\alpha}(x,n); q^2)$

where $p_{i,\alpha}(x,n) := \frac{1}{\alpha}\, q^{-2(N^-_{i-1}(x)-N^+_{i+1}(n))+\theta(2i-1)-1}$ and $c_{i,\alpha}(x,n) := \alpha q^{2(N^-_{i-1}(x)-N^+_{i+1}(n))+\theta(2i-1)+1}$.

Moreover, we have the following orthogonality relation:

$$\langle D_\alpha(\cdot,x), D_\alpha(\cdot,n)\rangle_{\mu_{\alpha,\sigma}} = \frac{\delta_{x,n}}{\mu_{\alpha,\sigma}(x)}$$

The associated symmetry is

$$S^{or}_{\alpha,\sigma} = BG_\alpha^{-1}\widehat{S^{tr}_\alpha} M \left(S^{tr}_\alpha\right)^t G_\alpha B^{-1}$$

where $M(x,n) = (-1)^{N(x)}\delta_{x,n}$ $\qquad B(x,n) = q^{N(x)}\delta_{x,n}$

From triangular to orthogonal dualities via scalar product

Theorem (Carinci, F., Groenevelt, 2021)

Let X be a Markov process on a countable state space Ω, with generator L. Let μ_1 and μ_2 be two reversible measures for X, and d_1, d_2, $\tilde{d}_1$ and $\tilde{d}_2$ be four self-duality functions for X. Suppose that

$$\langle d_1(x,\cdot), d_2(\cdot,n)\rangle_{\mu_1} = \frac{\delta_{x,n}}{\mu_2(n)} \qquad \text{and} \qquad \langle \tilde{d}_2(x,\cdot), \tilde{d}_1(\cdot,n)\rangle_{\mu_2} = \frac{\delta_{x,n}}{\mu_1(n)}$$

for $x, n \in \Omega$. Then the functions $D, \widetilde{D} : \Omega \times \Omega \to \mathbb{R}$ given by

$$D(x,n) := \langle \tilde{d}_1(x,\cdot), d_1(n,\cdot)\rangle_{\mu_1} \qquad \widetilde{D}(x,n) := \langle \tilde{d}_2(\cdot,x), d_2(\cdot,n)\rangle_{\mu_1}$$

Theorem (Cont.)

- *Then* D *and* $\widetilde{D}$ *are self-duality functions for* X*.*
- *They satisfy the biorthogonality relations:*

$$\langle D(\cdot, m), \widetilde{D}(\cdot, n)\rangle_{\mu_2} = \frac{\delta_{m,n}}{\mu_2(n)}, \qquad m, n \in \Omega.$$

- *In particular, if* $\widetilde{D} = c_1(x)c_2(n)D$*, where* c_1 *(resp.* c_2*) is a positive function of the total number of particles (resp. dual particles), then the biorthogonal equation becomes an orthogonality relation for* D *with respect to the weight* $c_1(x)\mu_2(x)$ *and with squared norm* $\frac{1}{c_2(n)\mu_2(n)}$*.*

Idea of the proof:

To get orthogonality from the biorthogonality is immediate.

The biorthogonal relation is a consequence of the scalar product relations of d_1 with d_2 and $\tilde{d}_1$ with $\tilde{d}_2$.

Finally the fact that D and $\widetilde{D}$ are self-duality by construction follows from the self-dualities of d_1, d_2, $\tilde{d}_1$ and $\tilde{d}_2$ and the self-adjointness of the generator L with respect to μ_1.

$q-$orthogonal polynomials as self-duality functions

We proved the following relations using the two triangular self-dualities for $\alpha, \beta \in \mathbb{R} \setminus \{0\}$

$$\langle D^{tr}_{1/\alpha q}(x,\cdot), \widehat{D}^{tr}_{-q/\beta}(\cdot,n)\rangle_{-\alpha} = \frac{\delta_{x,n}}{\mu_\beta(n)}$$

Computing two scalar products we find D and $\widetilde{D}$ biorthogonal self-duality functions. In particular,

$$D_\alpha(x,n) = \langle \widehat{D}^{\text{tr}}_{q/\alpha}(x,\cdot), D^{tr}_{1/\alpha q}(n,\cdot)\rangle_{-\alpha}$$
$$\widetilde{D}_{\alpha,\beta}(x,n) = \langle D^{tr}_{-1/\beta q}(\cdot,x), \widehat{D}^{tr}_{-q/\beta}(\cdot,n)\rangle_{-\alpha}$$

The computation leads to

$$D_\alpha(x,n) := \prod_{i=1}^{N} K_{n_i}(q^{-2x_i}; p_{i,\alpha}(x,n), \theta; q^2),$$

$$p_{i,\alpha}(x,n) := \frac{1}{\alpha}\, q^{-2(N_{i-1}^{-}(x)-N_{i+1}^{+}(n))+\theta(2i-1)-1}$$

and

$$\widetilde{D}_{\alpha,\beta}(x,n) := \frac{(\alpha q^{1+2N(x)-2\theta L-\theta})_\infty \;\; q^{N(x)(N(x)-1)}}{(\alpha q^{1-2N(n)-\theta})_\infty \;\; q^{N(n)(N(n)-1)}} \left(\frac{\alpha}{\beta}\right)^{N(x+n)} \cdot D_\alpha(x,n).$$

And so, by construction

$$\langle D_\alpha(\cdot,m), \widetilde{D}_{\alpha,\beta}(\cdot,n)\rangle_\beta = \frac{\delta_{m,n}}{\mu_\beta(n)}$$

Unitary symmetry

The unitary symmetry can be found from the scalar product used to find $D_\alpha(x, n)$, namely

$$D_\alpha(x, n) := \langle \widehat{D}^{\text{tr}}_{q/\alpha}(x, \cdot), D^{\text{tr}}_{1/\alpha q}(n, \cdot)\rangle_{-\alpha}$$

in matrix form it reads

$$D_\alpha = \widehat{D}^{\mathrm{tr}}_{q/\alpha} G^2_{-\alpha} (D^{\mathrm{tr}}_{1/\alpha q})^t = B G_\alpha^{-1} \widehat{S}^{\mathrm{tr}}_\alpha M (S^{\mathrm{tr}}_\alpha)^t G_\alpha^{-1} B^{-1}$$

Namely $S^{or} = B G_\alpha^{-1} \widehat{S}^{\mathrm{tr}}_\alpha M (S^{\mathrm{tr}}_\alpha)^t G_\alpha B^{-1}$.

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From the orthogonality of $D_\alpha \rightsquigarrow D_\alpha^t G_\alpha^2 D_\alpha = G_\alpha^{-2}$
we read the unitarity property of $S^{or} \rightsquigarrow (S^{or})^t\, G_\alpha^2 S^{or} = G_\alpha^2$

Applications

For $\sigma = -1$ the family of self-duality functions

$$\left\{ D_\alpha(\cdot, n),\ \ n \in \{0, 1, \ldots, \theta\}^N \right\}$$

form a basis for the L^2 space and so we can expand any function in terms of the self-duality polynomials: $f = \sum_n C_f(n) D_\alpha(\cdot, n)$.
Using the self-duality with one dual particle $n = \delta_i$:
$D_\alpha(x, \delta_i) = 1 - \frac{q^{2\theta i+1}}{(q^\theta - q^{-\theta})\,\alpha} \left[q^{-2N^-_{i-1}(x)} - q^{-2N^-_i(x)} \right]$ one sees that choosing $f = q^{-2N^-_i(x)}$, then

$$\begin{cases} C_f(0) = 1 - \alpha q^{-\theta-1} \left(1 - q^{-2\theta i}\right) \\ C_f(\delta_j) = \alpha q^{-1} \left(q^\theta - q^{-\theta}\right) q^{-2\theta j} 1_{\{j \le i\}} \\ C_f(n) = 0 \text{ for all n: } \|n\| > 1 \end{cases}$$

Space time correlations of the q^{-2}−exponential moments of $N_i^-(x)$

$$\mathbb{E}_{\mu_\alpha}\left[q^{-2N_i^-(x(s))}q^{-2N_j^-(x(t))}\right] =$$

$$\mathbb{E}_{\mu_\alpha}\left[\left(C_1(0)+\sum_{r=1}^{i}C_1(\delta_r)D_\alpha(x(s),\delta_r)\right)\left(C_2(0)+\sum_{k=1}^{j}C_2(\delta_k)D_\alpha(x(t),\delta_k)\right)\right] =$$

$$C_1(0)C_2(0)+C_1(0)\sum_{k=1}^{j}C_2(\delta_k)\mathbb{E}_{\mu_\alpha}D_\alpha(x(t),\delta_k)+C_2(0)\sum_{r=1}^{i}C_1(\delta_r)\mathbb{E}_{\mu_\alpha}D_\alpha(x(s),\delta_r)$$

$$+\sum_{r=1}^{i}\sum_{k=1}^{j}C_1(\delta_r)C_2(\delta_k)\mathbb{E}_{\mu_\alpha}\left[D_\alpha(x(s),\delta_r)D_\alpha(x(t),\delta_k)\right]$$

where $\mathbb{E}_{\mu_\alpha}\left[D_\alpha(x(s),\delta_r)D_\alpha(x(t),\delta_k)\right]$ can easily be computed using self-duality and the orthogonal property.

$$\mathbb{E}_{\mu_\alpha}\left[D_\alpha(x(s),\delta_r)D_\alpha(x(t),\delta_k)\right] =$$

$$\sum_x \mu_\alpha(x)\mathbb{E}_x\left[D_\alpha(x(s),\delta_r)D_\alpha(x(t),\delta_k)\right] =$$

$$\sum_x \mu_\alpha(x)\mathbb{E}_x\left[D_\alpha(x(s),\delta_r)\mathbb{E}_{x(s)}\left(D_\alpha(x(t-s),\delta_k)\right)\right] =$$

$$\sum_x \mu_\alpha(x)D_\alpha(x,\delta_r)\mathbb{E}_x\left(D_\alpha(x(t-s),\delta_k)\right) =$$

$$\sum_x \mu_\alpha(x)D_\alpha(x,\delta_r)\mathbb{E}_n\left(D_\alpha(x,\delta_{k(t-s)})\right) =$$

$$\mathbb{E}_n\left[\sum_x \mu_\alpha(x)D_\alpha(x,\delta_r)D_\alpha(x,\delta_{k(t-s)})\right] =$$

$$\frac{\mathbb{P}_n\left(k(t-s)=r\right)}{\mu_\alpha(r)}$$

$$\mathbb{E}_{\mu_\alpha}\left[D_\alpha(x(s),\delta_r)\right] = \sum_x \mu_\alpha(x)\mathbb{E}_x\left[D_\alpha(x(s),\delta_r)\right] =$$

$$\sum_x \mu_\alpha(x)\mathbb{E}_n\left[D_\alpha(x,\delta_{r(s)})\right] = \mathbb{E}_n\left[\sum_x \mu_\alpha(x) D_\alpha(x,\delta_{r(s)})\right] = 0$$

THANKS!