Orthogonal duality functions from Lie algebra representations

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Joint work with C.Franceschini, C.Giardinà

Outline

1. The symmetric inclusion process

2. Lie algebra approach to the symmetric inclusion process

3. Other processes

Stochastic duality

$X_i = \{\eta_i(t) \mid t \geq 0\}$, $i = 1, 2$, Markov process with state space Ω_i.

Definition: The processes X_1 and X_2 are in duality with duality function $D : \Omega_1 \times \Omega_2 \to \mathbb{C}$ if

$$\mathbb{E}_1\Big[D\big(\eta_1(t), \eta_2\big)\Big] = \mathbb{E}_2\Big[D\big(\eta_1, \eta_2(t)\big)\Big],$$

for all $t \geq 0$ and all starting configurations $\eta_1 \in \Omega_1$, $\eta_2 \in \Omega_2$.

If $X_1 = X_2$, the process is called self-dual.

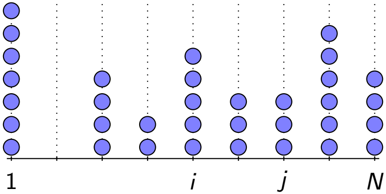
$L_i : F(\Omega_i) \to F(\Omega_i)$ generator for process X_i, i.e. $\exp(tL_i)$ is the semigroup corresponding to X_i.

Equivalent: X_1 and X_2 are in duality with duality function D, iff

$$\big[L_1 D(\,\cdot\,, \eta_2)\big](\eta_1) = \big[L_2 D(\eta_1, \,\cdot\,)\big](\eta_2), \qquad (\eta_1, \eta_2) \in \Omega_1 \times \Omega_2.$$

The symmetric inclusion process (SIP)

State space: $\Omega = \mathbb{N}_0^N$
x_i: number of particles on site i.
$k > 0$



Process generator

$$L^{\mathrm{SIP}} f(x) = \sum_{1 \le i < j \le N} x_i(2k + x_j)\left(f(x^{i,j}) - f(x)\right) + x_j(2k + x_i)\left(f(x^{j,i}) - f(x)\right)$$

Family of reversible measures for SIP:
Product of generalized negative binomial distributions

$$w_p(x) = \prod_{i=1}^{N} w_{k,p}(x_i), \qquad w_{k,p}(x) = \frac{(2k)_x}{x!} p^x (1-p)^{2k}, \; 0 < p < 1$$

Self-duality functions for SIP

Cheap self-duality function:

$$D_p^{ch}(x,y) = \prod_{i=1}^{N} \frac{x_i!}{(2k)_{x_i} p^{x_i}} \delta_{x_i,y_i}$$

Classical self-duality function:

$$D_p^{cl}(x,y) = \prod_{i=1}^{N} \frac{x_i! p^{-y_i}}{(y_i - x_i)!(2k)_{y_i}} \mathbb{1}_{\{y_i \le x_i\}}$$

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Orthogonal self-duality function [Franceschini, Giardinà; Redig, Sau]:

$$D_p^{or}(x,y) = \prod_{i=1}^{N} M_{x_i}(y_i; 2k, p)$$

where $M_x(y; 2k, p)$ is a Meixner polynomial.
Orthogonality relations:

$$\sum_{x \in \Omega} D_p^{or}(x,y) D_p^{or}(x,y') w_p(x) = 0 \qquad y \neq y'$$

The Lie algebra $\mathfrak{su}(1,1)$

Basis: H, E, F with commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

$*$**-structure:** $H^* = H, \quad E^* = -F, \quad F^* = -E.$

The **Casimir element** $\Omega = \frac{1}{2}H^2 + EF + FE \in \mathcal{U}(\mathfrak{su}(1,1))$ satisfies

- $[\Omega, X] = 0$ for all $X \in \mathfrak{su}(1,1)$
- $\Omega^* = \Omega$

A representation of $\mathfrak{su}(1,1)$

For $k > 0$, $0 < p < 1$, a $*$-representation on $\ell^2(\mathbb{N}_0, w_{k,p})$ with

$$w_{k,p}(x) = \frac{(2k)_x}{x!} p^x (1-p)^{2k}$$

is given by

$$\begin{aligned}
[\pi_{k,p}(H)f](x) &= 2(k+x)f(x), \\
[\pi_{k,p}(E)f](x) &= \frac{x}{\sqrt{p}} f(x-1), \\
[\pi_{k,p}(F)f](x) &= -\sqrt{p}\,(2k+x)f(x+1),
\end{aligned}$$

where $f(-1) = 0$.

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Eigenfunctions of $\pi_{k,p}(H)$: $\delta_y(x) := \delta_{x,y}$.

Relation with SIP

Coproduct: $\Delta : \mathcal{U}(\mathfrak{su}(1,1) \to \mathcal{U}(\mathfrak{su}(1,1) \otimes \mathcal{U}(\mathfrak{su}(1,1)$

$$\Delta(X) = 1 \otimes X + X \otimes 1, \qquad X \in \mathfrak{su}(1,1)$$

Coproduct of the Casimir:

$$\Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 + H \otimes H + 2F \otimes E + 2E \otimes F.$$

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Relation with generator L of SIP:
Recall:

$$Lf(x) = \sum_{1 \le i < j \le N} x_i(2k + x_j) \left(f(x^{i,j}) - f(x)\right) + x_j(2k + x_i) \left(f(x^{j,i}) - f(x)\right).$$

Denote $\pi_k = \pi_{k,p} \otimes \cdots \otimes \pi_{k,p}$ and let

$$Y = \frac{1}{2}\Big(1 \otimes \Omega + \Omega \otimes 1 - \Delta(\Omega)\Big),$$

then

$$L = \sum_{1 \le i < j \le N} \left[\pi_k(Y_{i,j}) + 2k^2\right]$$

Meixner polynomials

Definition:

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-x,-n\\ \beta\end{matrix};1-\frac{1}{c}\right)$$

Orthogonality relations:

$$\sum_{x=0}^{\infty} M_n(x)M_m(x)\frac{(\beta)_x}{x!}c^x = \delta_{m,n}\frac{n!}{(\beta)_n c^n(1-c)^\beta}, \qquad \beta>0,\ 0<c<1$$

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Three-term recurrence relation:

$$\frac{c-1}{\sqrt{c}}(\tfrac{1}{2}\beta + x)M_n(x) = \sqrt{c}(\beta+n)M_{n+1}(x) - \frac{c+1}{\sqrt{c}}(\tfrac{1}{2}\beta + n)M_n(x) + \frac{n}{\sqrt{c}}M_{n-1}(x)$$

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Self-duality property: $M_n(x;\beta,c) = M_x(n;\beta,c)$

Meixner polynomials as eigenfunctions

Define

$$X_p = -\frac{1+p}{2\sqrt{p}} H + E - F \in \mathfrak{su}(1,1),$$

then

- $X_p^* = X_p$
- $\mathfrak{su}(1,1)$ is generated by H and X_p

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Action of X_p:

$$[\pi_{k,p}(X_p)f](x) = \sqrt{p}(2k+x)f(x+1) - \frac{1+p}{\sqrt{p}}(k+x)f(x) + \frac{x}{\sqrt{p}}f(x-1).$$

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Eigenfunctions of $\pi_{k,p}(X_p)$: For $M(x,y) = M_x(y; 2k, p)$

$$[\pi_{k,p}(X_p) M(\cdot, y)](x) = \frac{p-1}{\sqrt{p}}(k+y)\, M(x,y), \qquad y \in \mathbb{N}_0.$$

Meixner polynomials as intertwining functions

Define $H_p = \frac{2\sqrt{p}}{p-1} X_p$, then

$$
\begin{aligned}
&[\pi_{k,p}(H_p)M(\,\cdot\,,y)](x) \\
&= -\frac{2p}{1-p}(2k+x)M(x+1,y) + \frac{1+p}{1-p}2(k+x)M(x,y) - \frac{2x}{1-p}M(x-1,y) \\
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From difference equation for Meixner polynomials

$$\begin{aligned}
&\big[\pi_{k,p}(H)M(\cdot,y)\big](x) \\
&= 2(k+x)M(x,y) \\
&= -\frac{2p}{1-p}(2k+y)M(x,y+1) + \frac{1+p}{1-p}2(k+y)M(x,y) - \frac{2y}{1-p}M(x,y-1).
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$$

So we have the intertwining relation

$$
\begin{aligned}
[\pi_{k,p}(H)M(\,\cdot\,,y)](x) &= [\pi_{k,p}(H_p)M(x,\,\cdot\,)](y) \\
[\pi_{k,p}(H_p)M(\,\cdot\,,y)](x) &= [\pi_{k,p}(H)M(x,\,\cdot\,)](y)
\end{aligned}
$$

Meixner polynomials as intertwining functions

Define an isomorphism $\theta : \mathcal{U}(\mathfrak{su}(1,1)) \to \mathcal{U}(\mathfrak{su}(1,1))$ by

$$\theta(H) = H_p \qquad \text{and} \qquad \theta(H_p) = H.$$

Theorem: For all $X \in \mathcal{U}(\mathfrak{su}(1,1))$

$$[\pi_{k,p}(\theta(X)^*)M(\,\cdot\,,y)](x) = [\pi_{k,p}(X)M(x,\,\cdot\,)](y), \qquad x,y \in \mathbb{N}_0.$$

Remark: $\Lambda : \ell^2(\mathbb{N}_0, w_p) \to \ell^2(\mathbb{N}_0, w_p)$ given by

$$\Lambda f(y) = \sum_{x\in\mathbb{N}_0} f(x)M(x,y)w_{k,p}(x)$$

defines a unitary intertwiner between $\pi_{k,p}$ and $\pi_{k,p} \circ \theta$.

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Corollary: For all $X \in \mathcal{U}(\mathfrak{su}(1,1)) \otimes \mathcal{U}(\mathfrak{su}(1,1))$

$$[\pi_{k,p} \otimes \pi_{k,p}(\theta \otimes \theta(X^*))M(\,\cdot\,,y_1)M(\,\cdot\,,y_2)](x_1,x_2) = [\pi_{k,p} \otimes \pi_{k,p}(X)M(x_1,\,\cdot\,)M(x_2,\,\cdot\,)](y_1,y_2).$$

Self-duality functions for SIP

Ω in terms of H and H_p:

$$\Omega = -\frac{(p-1)^2}{8p}(H^2 + H_p^2) + \frac{1-p^2}{8p}(HH_p + H_pH) + \frac{(p-1)^2}{32p}[H, H_p]^2.$$

So $\theta \otimes \theta(\Delta(\Omega)) = \Delta(\Omega)$.

Theorem: $\prod_{i=1}^{N} M(x_i, y_i)$ is a self-duality function for $\pi_k(\Delta(\Omega)_{i,j})$, hence for SIP.

Stochastic self-duality and self-dual special functions

- A and B operators on $F(\Omega)$
- $f : \Omega \times \Omega \to \mathbb{C}$ is an intertwining function for operators A and B

$$[Af(\,\cdot\,, y)](x) = [Bf(x,\,\cdot\,)](x)$$

- $f(x, y) = f(y, x)$ (self-duality as a function)

Theorem:
If $L(A, B)$ is a linear combination of words in A and B such that $L(A, B) = L(B, A)$, then f is a self-duality function for L.

General setup for orthogonal duality

- $\mathfrak{g}$ Lie algebra with $*$-structure
- π_1, π_2 unitarily equivalent $*$-representations of $\mathcal{U}(\mathfrak{g})^{\otimes N}$ on $L^2(\Omega_i, \mu_i)$
- $\Lambda : L^2(\Omega_1, \mu_1) \to L^2(\Omega_2, \mu_2)$ unitary intertwiner given by

$$\Lambda f(y) = \int_{\Omega_1} f(x) K(x,y) d\mu_1(x),$$

with kernel $K : \Omega_1 \times \Omega_2 \to \mathbb{C}$ satisfying

$$[\pi_1(X)K(\,\cdot\,, y)](x) = [\pi_2(X^*)K(x, \,\cdot\,)](y), \quad (x,y) \in \Omega_1 \times \Omega_2, \; X \in \mathcal{U}(\mathfrak{g})^{\otimes N}$$

Let $Y \in \mathcal{U}(\mathfrak{g})^{\otimes N}$ such that $Y^* = Y$ and $\pi_1(Y)$, $\pi_2(Y)$ are generators for Markov processes X_1 and X_2 respectively, then X_1 is dual to X_2 with duality function K.

If Ω_1 is countable, then

$$\int_{\Omega_2} K(x,y) K(x', y) d\mu_2(y) = 0, \qquad x \neq x'.$$

Laguerre polynomials

Consider

$$X_1 = -H + E - F.$$

Eigenfunctions: The Laguerre polynomials $L(x, y) = \frac{x! p^{-x/2}}{(2k)_x} L_x^{(2k-1)}(y)$ satisfy

$$[\pi_{k,p}(X_1) L(\cdot\,, y)](x) = -y L(x, y), \qquad x \in \mathbb{N}_0,\ y \in \mathbb{R}^+.$$

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Action of H: from differential equation for Laguerre polynomials

$$\begin{aligned}[\pi_k(H)L(\cdot,y)](x) &= 2(k+x)L(x,y) \\ &= -2y\frac{\partial^2}{\partial y^2}L(x,y) - 2(2k-y)\frac{\partial}{\partial y}L(x,y) + 2k\,L(x,y)\end{aligned}$$

Another representation of $\mathfrak{su}(1,1)$

Representation in terms of differential operators:

$$[\sigma_k(H)f](x) = -2x\frac{\partial}{\partial x}f(x) - (2k-x)f(x),$$
$$[\sigma_k(E)f](x) = -\frac{1}{2}ixf(x),$$
$$[\sigma_k(F)f](x) = -2ix\frac{\partial^2}{\partial x^2}f(x) - 2i(2k-x)\frac{\partial}{\partial x}f(x) + \frac{i}{2}(4k-x)f(x).$$

Intertwining relation: for all $X \in \mathcal{U}(\mathfrak{su}(1,1))$

$$[\pi_{k,p}(\theta(X)^*)L(\cdot\,,y)](x) = [\sigma_k(X)L(x,\cdot\,)](y)$$

where θ is defined by $\theta(F-E) = iH$ and $\theta(E) = \frac{i}{2}X_1$.
Unitary intertwiner: $\Lambda : \ell^2(\mathbb{N}_0, w_{k,p}) \to L^2(\mathbb{R}^+, w(x;k))$ with

$$w(x;k) = \frac{x^{2k-1}e^{-x}}{\Gamma(2k)}$$

defined by

$$\Lambda f(y) = \sum_{x\in\mathbb{N}_0} f(x)L(x,y)w_{k,p}(x)$$

Duality between SIP en BEP

Apply the intertwining relation with $\Delta(\Omega)$:

Theorem: SIP is dual to the Brownian Energy Process with generator

$$
\begin{aligned}
L^{BEP} &= \sum_{1 \le i < j \le N} \sigma_k(Y_{i,j}) + 2k^2 \\
&= \sum_{1 \le i < j \le N} y_i y_j \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right)^2 - 2k(y_i - y_j) \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right),
\end{aligned}
$$

with duality function $\prod_{i=1}^N L(x, y)$.

Self-duality of BEP

Eigenfunctions of $\sigma_k(F)$: For $J(x,y) = e^{\frac{1}{2}(x+y)}(xy)^{-k+\frac{1}{2}}J_{2k-1}(\sqrt{xy})$

$$\begin{aligned}[\sigma_k(F)J(\,\cdot\,,y)](x) &= \left[\left(-2ix\frac{\partial^2}{\partial x^2} - 2i(2k-x)\frac{\partial}{\partial x} + \frac{i}{2}(4k-x)\right)J(\,\cdot\,,y)\right](x) \\ &= \frac{1}{2}iyJ(x,y) \\ &= [\sigma_k(E^\star)J(x,\,\cdot\,)](y).\end{aligned}$$

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Intertwining property: $[\sigma_k(X^\star)J(\,\cdot\,,y)](x) = [\sigma_k(X)J(x,\,\cdot\,)](x)$
Unitarity of corresponding intertwiner corresponds to unitarity of the Hankel transform.

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Theorem: BEP is self-dual with duality function $\prod_{i=1}^{N} J(x_i, y_i)$.

Orthogonal dualities from Lie algebras

- From $\mathfrak{su}(1,1)$:
  - Self-duality for SIP; Meixner polynomials
  - Duality between SIP and BEP; Laguerre polynomials
  - Self-duality for BEP; Hankel transform

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- From $\mathfrak{su}(2)$:
  - Self-duality for the Symmetric Exclusion Process on N sites; Krawtchouck polynomials

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- From $\mathfrak{su}(2)$:
  - Self-duality for the Symmetric Exclusion Process on N sites; Krawtchouck polynomials
- From the Heisenberg Lie algebra:
  - Self-duality for the Indendent Random Walkers (IRW) on N sites; Charlier polynomials
  - Duality between IRW and a diffusion process; Hermite polynomials
  - Self-duality for the diffusion process; Fourier transform

Outlook

Extend method for:

- Asymmetric processes (ASEP and ASIP): $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{su}(1,1))$
- Multispecies processes: higher rank Lie algebras en quantum algebras