Duality in the continuum

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Outline

1) Recap of basic dualities in SIP, SEP, IRW
2) Self-Duality for IRW revisited
3) Self-Duality for general independent processes
    3.1) Factorial moment dualities
    3.2) Orthogonal (Charlier) dualities
4) Extension to consistent systems
5) Examples

Recap on dualities

- Let V be a finite set and $\Omega = \mathbb{N}^V$ denote the configuration space.
- The Markov process with state space Ω defined via its generator

$$Lf(\eta) = \sum_{i,j \in V} p(i,j)\eta_i(\alpha + \sigma\eta_j)\left(f(\eta - \delta_i + \delta_j) - f(\eta)\right)$$

(with $p(i,j) = p(j,i)$ a symmetric edge weight) is called

1. SIP(α) when $\alpha > 0$ and $\sigma = +1$
2. SEP(α) when $\alpha \in \mathbb{N}$, $\alpha > 0$ and $\sigma = -1$
3. IRW(α) when $\sigma = 0$.

- These processes are (very) special, because they are self-dual, i.e., there exist functions $D : \Omega \times \Omega \to \mathbb{R}$ such that

$$\mathbb{E}_\xi D(\xi(t), \eta) = \mathbb{E}_\eta D(\xi, \eta(t))$$

- We are mostly interested in duality functions which (homogeneously) factorize over vertices, i.e.,

$$D(\xi, \eta) = \prod_{i \in V} d(\xi_i, \eta_i)$$

- The reversible measures of these processes are products of:
  1. SIP: Negative binomials (discrete Gamma distributions): $\nu_p^{(\alpha)}(n) = \frac{p^n \Gamma(\alpha+n)}{n! \Gamma(\alpha)}(1-p)^\alpha$, $0 < p < 1$.
  2. SEP: Binomials: $\nu_p(n) = \binom{\alpha}{n}(1-p)^{\alpha-n} p^n$, $0 < p < 1$, $n \in \{0, \ldots, \alpha\}$.
  3. IRW: Poisson: $\nu_\theta = \frac{\theta^n}{n!} e^{-\theta}$.

There are two different factorizing self-duality functions of special interest

a) (modified) Factorial moment self-duality functions (classical duality functions)

$$d(k,n) = \frac{n!}{(n-k)!}\frac{1}{m(\alpha,k)}$$

with

$$m(\alpha,k) = \begin{cases} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, & \text{for SIP} \\ \frac{\alpha!}{(\alpha-k)!}, & \text{for SEP} \\ 1, & \text{for IRW} \end{cases}$$

$\frac{m(\alpha,k)}{k!}$ is a reversible weight for the dynamics, i.e.,

$$\prod_i \frac{m(\alpha,\xi_i)}{\xi_i!}$$

satisfies detailed balance.

b) Orthogonal polynomial self-duality functions $\mathcal{D}_\theta(\xi, \eta)$
   1) IRW: products of Charlier polynomials.
   2) SIP: products of Meixner polynomials.
   3) SEP: products of Krawtchouk polynomials.

The following relations hold between the reversible measures and the duality functions.

1. Classical duality functions

$$\int D(\xi, \eta)\nu_\theta(d\eta) = \rho(\theta)^{|\xi|}$$

   with $\rho(\theta) = \int D(\delta_x, \eta)\nu_\theta(d\eta)$
2. Orthogonal duality functions

$$\int \mathcal{D}_\theta(\xi, \eta)\mathcal{D}_\theta(\xi', \eta)\nu_\theta(d\eta) = \delta_{\xi,\xi'}\psi(\theta, \xi)$$

   where $\psi(\theta, \xi)$ satisfies "inverse" detailed balance, i.e.,

$$\psi(\theta, \xi')p_t(\xi, \xi') = \psi(\theta, \xi)p_t(\xi', \xi)$$

Relation between classical dualities and orthogonal dualities for the case IRW (Charlier case).

For $\xi = \sum_{i=1}^{n} \delta_{y_i}$ we have

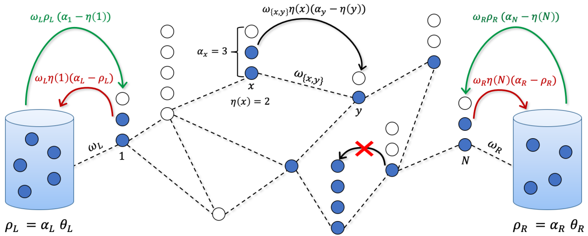
$$\begin{aligned} \mathcal{D}_\theta(\xi,\eta) &= \sum_{\xi'\leq\xi} (-\theta)^{|\xi|-|\xi'|} \binom{\xi}{\xi'} D(\xi',\eta) \\ &= \sum_{I\subset[n]} (-\theta)^{n-|I|} D\left(\sum_{i\in I} \delta_{y_i}, \eta\right) \end{aligned}$$

With $[n] = \{1,\ldots,n\}$.

Use of orthogonal dualities

a) Orthogonal polynomial dualities are very useful in the context of studying cumulants in non-equilibrium steady states (NESS) (joint work S. Floreani, F. Sau). In systems with two reservoir parameters ρ_L, ρ_R, cumulants of order n in the NESS are of equal to $(\rho_R - \rho_L)^n$ multiplied with a function that is not depending on the reservoir parameters.

$$\mathbb{E}\left(\prod_{i=1}^{n}(\eta_{x_i} - \rho_i)\right) = (\rho_R - \rho_L)^n \psi(x_1, \ldots, x_n)$$



Use of orthogonal dualities

b) Orthogonal polynomial dualities are useful in the study of fluctuation fields. For $V = \mathbb{Z}^d$, we define the n-th order fields

$$\begin{aligned} & X_n(N, \eta, \phi, t) \\ = \; & \frac{1}{N^{nd/2}} \sum_{x_1, \ldots, x_n \in \mathbb{Z}^d} \mathcal{D}_\theta \left(\sum_{i=1}^{n} \delta_{x_i}, \eta(N^2 t) \right) \phi\left(\frac{x_1}{N}, \ldots, \frac{x_n}{N} \right) \end{aligned}$$

with ϕ a symmetric test function.

- This field for $n = 1$ is the density fluctuation field.
- The limit of these fields (cf. M. Ayala and G. Carinci. R, EJP 2021) yield a recursively defined martingale problem generalizing the (distribution-valued) Ornstein-Uhlenbeck process for $n = 1$, and the second order field defined for SEP by Assing (2007) and Goncalves, Jara (2019).
- Chen and Sau (2020) study fields of classical duality functions on the hydrodynamic scale (higher-order hydrodynamics).

Use of orthogonal dualities

c) Orthogonal polynomial dualities in the study of quantitative Boltzmann-Gibbs principles, which provide a systematic expansion of fluctuation fields of general local functions f

$$Y(N, \eta, \phi, f, t) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{Z}^d} \phi(\tfrac{x}{N}) \tau_x f(\eta(N^2 t))$$

(for f a local function of ν_θ expectation zero) in terms of orthogonal self-duality polynomials.

$$Y(N, \eta, \phi, f, t) \approx c(f, \theta) \frac{1}{N^{d/2}} \sum_{x \in \mathbb{Z}^d} \phi(\tfrac{x}{N})(\eta_x(N^2 t) - \mathbb{E}_{\nu_\theta}(\eta_x(N^2 t)))$$

where $a \approx b$ means $\| \int_0^T (a - b) \|_{L_2} \to 0$.

Main questions

- What remains of these dualities when V is no longer discrete, is e.g. $\mathbb{R}^d$ with independent or interacting Brownian motions?
- Is it possible to obtain dualities for the SPDE's that arise as limits of particle systems?

Strategy

- "Notationally lift" the discrete dualities such that they start to make sense for V any (complete, separable) metric space.
- Start with independent walkers and then see where independence is really used, replace it by something weaker (consistency)whenever possible.

Self-duality for IRW revisited

Let us first consider one dual particle. Then the self-duality reads

$$\mathbb{E}_\eta \eta_x(t) = \sum_y p_t(x,y)\eta_y = \mathbb{E}_{\delta_x}\left(D(\delta_{X(t)},\eta)\right)$$

with $p_t(x,y)$ the transition probability of a single walker. Let us denote by $\mathcal{X} = \mathcal{X}_i, i = 1, \ldots, M$ a labeled initial configuration of the independent walkers.
The connection with the configuration reads

$$\eta = \sum_i \delta_{\mathcal{X}_i}$$

to be read as a measure on V with the obvious identification

$$\eta(\{x\}) = \eta_x = \left(\sum_i \delta_{\mathcal{X}_i}\right)(\{x\}) = \sum_i I(\mathcal{X}_i = x)$$

With this notation, we can do the following simple computation

$$\begin{aligned}
\mathbb{E}_\eta\left(\sum_x \eta_x(t)\delta_x\right) &= \mathbb{E}_{\mathcal{X}}\left(\sum_i \delta_{\mathcal{X}_i(t)}\right)\\
&=^* \sum_i \mathbb{E}_{\mathcal{X}_i(0)}(\delta_{\mathcal{X}_i(t)})\\
&= \sum_i \left(\sum_x p_t(\mathcal{X}_i(0), x)\delta_x\right)\\
&=^{**} \sum_x \left(\sum_i p_t(x, \mathcal{X}_i(0))\right)\delta_x\\
&= \sum_x \left(\sum_y p_t(x, y)\eta_y\right)\delta_x
\end{aligned}$$

In $*$ we used independence, and in $**$ we used symmetry.

Identifying the terms with δ_x gives

$$\mathbb{E}_\eta \eta_x(t) = \sum_y p_t(x,y)\eta_y$$

What we obtained can now be rewritten as

$$(\mathbb{E}_\eta(\eta_t))[dx] = \left(\int p_t(x,y)\eta(dy) \right) \lambda(dx) \tag{1}$$

with λ the counting measure. This is a candidate identity which is true in general, i.e. for independent reversible processes on a general Polish space V. Another way of writing (1) is

$$\frac{d(\mathbb{E}_\eta(\eta_t))}{d\lambda}(x) = \int p_t(x,y)\eta(dy)$$

With n dual particles the situation is very similar. First a simple combinatorial fact

$$\sum_{i_1,\ldots,i_n}^{\neq} \delta_{(\mathcal{X}_{i_1},\ldots,\mathcal{X}_{i_n})}(\{(y_1,\ldots,y_n)\}) = D\left(\sum_{i=1}^{n}\delta_{y_i},\eta\right)$$

In other words

$$\sum_{i_1,\ldots,i_n}^{\neq} \delta_{(\mathcal{X}_{i_1},\ldots,\mathcal{X}_{i_n})} = \sum_{y_1,\ldots y_n} \delta_{(y_1,\ldots,y_n)} D\left(\sum_{i=1}^{n}\delta_{y_i},\eta\right)$$

e.g.

$$\sum_{i_1,i_2}^{\neq} \delta_{(\mathcal{X}_{i_1},\mathcal{X}_{i_2})}\{(x,y)\} = \begin{cases}\eta_x\eta_y \text{ if } x\neq y\\ \eta_x(\eta_x-1) \text{ if } x=y\end{cases}$$

The same type of computation can then be done for

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{X}}\left(\sum_{i_1,\ldots,i_n}^{\neq} \delta_{(\mathcal{X}_{i_1}(t),\ldots,\mathcal{X}_{i_n}(t))}\right) \\
= &\sum_{i_1,\ldots,i_n}^{\neq} \mathbb{E}_{(\mathcal{X}_{i_1}(0),\ldots,\mathcal{X}_{i_n}(0))}\left(\delta_{(\mathcal{X}_{i_1}(t),\ldots,\mathcal{X}_{i_n}(t))}\right) \\
= &\sum_{y_1,\ldots,y_n}\sum_{i_1,\ldots,i_n}^{\neq} p_t^{(n)}((y_1,\ldots,y_n);(\mathcal{X}_{i_1}(0),\ldots,\mathcal{X}_{i_n}(0)))\left(\delta_{(y_1,\ldots y_n)}\right)
\end{aligned}
$$

Here once more, in the first step we used independence, and in the second step symmetry. Identifying the terms with $\delta_{(y_1,\ldots,y_n)}$ yields then the IRW self-duality

$$
\mathbb{E}_\eta D\left(\sum_{i=1}^n \delta_{y_i}, \eta(t)\right) = \mathbb{E}_\xi D(\xi(t),\eta)
$$

with $\xi = \sum_{i=1}^n \delta_{y_i}$.

Indeed

$$
\begin{aligned}
& \mathbb{E}_{\mathcal{X}} \left(\sum_{i_1,\ldots,i_n}^{\neq} \delta_{(\mathcal{X}_{i_1}(t),\ldots,\mathcal{X}_{i_n}(t))} \right) \\
= & \sum_{y_1,\ldots,y_n} \delta_{(y_1,\ldots,y_n)} \mathbb{E}_\eta D(\sum_{i=1}^n \delta_{y_i}, \eta(t)) \\
= & \sum_{y_1,\ldots,y_n} \delta_{(y_1,\ldots y_n)} p_t^{(n)}((y_1,\ldots,y_n);(\mathcal{X}_{i_1}(0),\ldots,\mathcal{X}_{i_n}(0))) \\
= & \sum_{y_1,\ldots,y_n} \delta_{(y_1,\ldots y_n)} \sum_{z_1,\ldots,z_n} p_t^{(n)}((y_1,\ldots,y_n);(z_1,\ldots z_n)) D(\sum_{i=1}^n \delta_{z_i}, \eta)
\end{aligned}
$$

which upon identifying the terms with $\delta_{(y_1,\ldots,y_n)}$ gives

$$
\mathbb{E}_\eta D(\sum_{i=1}^n \delta_{y_i}, \eta(t)) = \mathbb{E}_{(y_1,\ldots,y_n)} D(\sum_{i=1}^n \delta_{y_i(t)}, \eta)
$$

Let us now put this in natural notation which then makes sense in any Polish space V. For a point measure η (of the form $\sum_i \delta_{\mathcal{X}_i}$) the n-th factorial measure is defined as

$$\eta^{(n)}(dx_1, \dots dx_n) = \sum_{i_1, \dots, i_n}^{\neq} \delta_{(\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_n})}(dx_1, \dots dx_n)$$

which is a (possibly random) measure on V^n. Now assume that we have independent Markov processes on V with reversible measure λ and assume that the transition probabilities read

$$p_t(x, dy) = p_t(x, y)\lambda(dy)$$

By the reversibility of λ we have $p_t(x, y) = p_t(y, x)$. Then we have the following self-duality result

$$\frac{d\mathbb{E}_\eta[\eta_t^{(n)}]}{d\lambda^{\otimes n}}(y_1, \dots, y_n) = \int \prod_{i=1}^n p_t(y_i, z_i)\eta^{(n)}(dz_1, \dots, dz_n)$$

Independent particles: general results

We now consider independent Markov processes on V (Polish space) and call $\eta(t) = \sum_i \delta_{\mathcal{X}_i(t)}$ the corresponding configuration process. Then we have

- Intertwining for general processes

$$\mathbb{E}_\eta J_n(f_n, \eta(t)) = P_t J_n(f_n, \eta) = J_n(p_t^{(n)} f_n, \eta)$$

  where

$$J_n(f_n, \eta) = \int f_n(x_1, \ldots x_n) \eta^{(n)}(dx_1, \ldots, dx_n)$$

  where P_t is the semigroup acting on the configuration variable, and p_t^n is n-particle semigroup.

- Duality for reversible processes: if $p(x, dy) = p_t(x, y)\lambda(dy)$ with λ a reversible measure then this intertwining leads to the "self-duality":

$$\frac{d\mathbb{E}_\eta[\eta_t^{(n)}]}{d\lambda^{\otimes n}}(y_1, \ldots, y_n) = J_n\left(\prod_{i=1}^n p_t(y_i, \cdot), \eta\right).$$

In order re-connect with the discrete case, one has to choose special f_n For $B_1, \ldots, B_N$ disjoint Borel subsets of V, define

$$f_n = 1_{B_1}^{\otimes d_1} \otimes \ldots \otimes 1_{B_N}^{\otimes d_N}$$

with $d_1 + \ldots + d_N = n$. then

$$\frac{1}{n!} J_n(f_n, \eta) = \prod_{i=1}^{N} \binom{\eta(B_i)}{d_i} I(d_i \le \eta(B_i))$$

which corresponds to choosing d_i dual particles in the set B_i, $i = 1, \ldots, N$.

Generalization to consistent processes

- We have used

$$\mathbb{E}_{\mathcal{X}}\left(\delta_{(X_{i_1}(t),\ldots,\mathcal{X}_{i_{n(t)}})}\right) = \mathbb{E}_{(X_{i_1}(0),\ldots\mathcal{X}_{i_n}(0))}\left(\delta_{(X_{i_1}(t),\ldots\mathcal{X}_{i_{n(t)}})}\right)$$

which we based on independence.
- This can be generalized to processes satisfying consistency (cf. Carinci, Giardinà, R (2019); Kipnis, Marchioro, Presutti (1986); Le Jan, Raimond (2004); Schertzer, Sun, Swart (2017); Howitt Warren (2009); Brockington, Warren (2021)) .

Forms of consistency.

a) Strong consistency: the k dimensional marginals of any n tuple evolved in time equals k-particle-process. I.e., for all n, and $t \geq 0$, $x_1, \ldots, x_n \in V^n$, and for $I = \{i_1, \ldots, i_k\} \subset [n]$

$$(X_1^{x_1}(t), \ldots, X_n^{x_n}(t))_I = (X_1^{x_{i_1}}(t), \ldots, X_k^{x_{i_k}}(t)) \qquad (2)$$

where the equality is in distribution.

b) Consistency: the configuration dynamics (i.e., the labeled process modulo permutations) commutes with the operation of taking away a random particle from the configuration. This can be shown to be equivalent with the commutation property

$$[P_t, \mathcal{A}] = 0$$

where $\mathcal{A}$ is the annihilation operator

$$\mathcal{A}f(\eta) = \int f(\eta - \delta_x)\eta(dx)$$

which in the discrete case reads

$$\mathcal{A}f(\eta) = \sum_x \eta_x f(\eta - \delta_x)$$

Examples of consistent processes

a) IRW, SIP, SEP on discrete sets V.

b) General independent processes.

c) Howitt-Warren flow of sticky Brownian motions (Brockington, Warren (2021)).

d) Generalized inclusion process: process defined on point measures on V defined as follows. Let α denote finite measure on V, $p : V \times V \to \mathbb{R}$ bounded symmetric and measurable.

$$Lf(\eta) = \int \int p(x,y)(f(\eta - \delta_x + \delta_y) - f(\eta))(\alpha + \eta)[dy]\eta[dx]$$

Reversible measures for this process are the point measure analogue of discrete Gamma distributions, i.e., for $0 < p < 1$ and mutually disjoint measurable subsets of V, $\eta(A_1), \ldots, \eta(A_n)$ are independent with

$$\mathbb{P}\left(\eta(A) = k\right) = \frac{\Gamma(\alpha(A) + k)}{\Gamma(\alpha(A))} \frac{p^k}{k!} (1 - p)^{\alpha(A)}$$

Orthogonal dualities: independent case

Let us first see how to obtain directly the orthogonal self-duality with self-duality functions (for independent symmetric random walks on the finite set V). For $\xi = \sum_{i=1}^n \delta_{y_i}$:

$$\begin{aligned}\mathcal{D}_\theta(\xi,\eta) &= \sum_{\xi'\le\xi} (-\theta)^{|\xi|-|\xi'|}\binom{\xi}{\xi'} D(\xi',\eta) \\ &= \sum_{I\subset[n]} (-\theta)^{n-|I|} D\left(\sum_{i\in I}\delta_{y_i},\eta\right)\end{aligned}$$

$$\begin{aligned}\mathbb{E}^{\mathrm{IRW}}_\eta[\mathcal{D}_\theta(\xi,\eta(t))] &= \sum_{r=0}^n (-\theta)^{n-r} \sum_{I\subset[n]:|I|=r} \mathbb{E}^{\mathrm{IRW}}_{(y_1,\ldots,y_n)_I}[D(\sum_{i=1}^r \delta_{Y_t(i)},\eta)] \\ &= \mathbb{E}^{\mathrm{IRW}}_{(y_1,\ldots,y_n)}[\sum_{r=0}^n (-\theta)^{n-r} \sum_{I\subset[n]:|I|=r} D(\sum_{i\in I}\delta_{Y_t(i)},\eta)] \\ &= \mathbb{E}^{\mathrm{IRW}}_{(y_1,\ldots,y_n)}[D_\theta(\xi(t),\eta)] = \mathbb{E}^{\mathrm{IRW}}_\xi[\mathcal{D}_\theta(\xi(t),\eta)],\end{aligned}$$

Moving towards the general setting, we have to rewrite $\mathcal{D}_\theta$ in an appropriate way. To do so, introduce the orthogonal factorial measure with parameter θ

$$\begin{aligned} & \eta_t^{(n),\theta}(dx_1,\ldots,dx_n)) \\ = & \sum_{r=0}^{n}(-\theta)^{n-r}\sum_{I\subset[n]:|I|=r}\eta_t^{(r)}(d(x_1,\ldots,x_n)_I)\otimes\lambda^{n-r}(d(x_1,\ldots,x_n)_{[n]\setminus I}) \end{aligned}$$

We have

$$\eta^{(n),\theta}(\{(y_1,\ldots,y_n)\})=\mathcal{D}_\theta\left(\sum_{i=1}^{n}\delta_{y_i},\eta\right)$$

As a consequence, the self-duality with $\mathcal{D}_\theta$ can be restated equivalently as follows

$$\frac{d\mathbb{E}_\eta(\eta_t^{(n),\theta})}{d\lambda^{\otimes n}}(y_1,\ldots,y_n)=\int\prod_{i=1}^{n}p_t(y_i,z_i)\eta^{(n),\theta}(dz_1\ldots dz_n)$$

which is a statement making sense in the general setting.

We then obtain two general statements for independent particles on a general (Polish) space V

1. Intertwining for general independent processes

$$\mathbb{E}_\eta I_{n,\theta}(f_n, \eta_t) = I_{n,\theta}(p_t^{(n)} f_n, \eta)$$

with

$$I_{n,\theta}(f_n, \eta) = \int f_n(x_1, \ldots, x_n)\eta^{(n),\theta}(dx_1 \ldots dx_n)$$

2. Self-duality in the reversible case

$$\begin{aligned} &\mathbb{E}_\eta(\eta^{(n),\theta}(t))[dy_1, \ldots dy_n] \\ = &\left(\prod_{i=1}^{n} p_t(y_i, z_i)\eta^{(n),\theta}(dz_1 \ldots dz_n)\right)\lambda(dy_1)\ldots\lambda(dy_n) \end{aligned}$$

Choosing once more, for $B_1, \ldots, B_N$ disjoint Borel subsets of V,

$$f_n = 1_{B_1}^{\otimes d_1} \otimes \ldots \otimes 1_{B_N}^{\otimes d_N}$$

with $d_1 + \ldots + d_N = n$. then, with $\theta = 1$, and $I_{n,1} = I_n$ we get

$$\frac{1}{n!} I_n(f_n, \eta) = \prod_{k=1}^{N} \lambda(B_k)^{d_k} \mathcal{C}_{d_k}(\eta(B_k); \lambda(B_k))$$

where

$$\mathcal{C}_n(x, \alpha) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \alpha^{-k} (x)_k$$

is the Charlier polyomial of order n.
(θ of previous slide is now included in λ).

Relation with Wiener chaos decomposition

$$\int_0^t C_n(N_{s-}, s)d(N_s - s) = \frac{C_{n+1}(N_t, t)}{n+1}$$

$$\alpha^n C_n(\cdot, \alpha) = \delta^n 1$$

with

$$\delta f(n) = nf(n-1) - \alpha f(n)$$

More generally for $\phi : V \to \mathbb{R}$

$$\delta(\phi)f(\eta) = \int \phi(x)f(\eta - \delta_x)\eta(dx) - \int \phi(x)\lambda(dx)$$

Then

$$I_n(\phi_1 \otimes \ldots \otimes \phi_n, \eta) = [\delta(\phi_1) \otimes \ldots \otimes \delta(\phi_n)]1(\eta)$$

The operators δ are natural objects appearing in "Poissonian white noise calculus", Poissonian Wiener chaos expansion, Poissonian Malliavin calculus.

Orthogonal dualities: the consistent case

The extension to the consistent case requires a “compatibility” assumption between the reversible measures $\lambda^{(n)}$ and $\lambda^{(r)}$, for $n > r$. We assume that there exists a kernel $k_{n,r}$ such that $\lambda^{(n)} = \lambda^{(r)} \otimes k_{n,r}$, i.e.,

$$\int f(x_1, \ldots, x_n)\lambda^{(n)}(dx_1 \ldots dx_n) =$$

$$\int f(x_1, \ldots, x_n) k_{n,r}(x_1, \ldots, x_r; dx_{r+1} \ldots dx_n)\lambda^{(r)}(dx_1 \ldots dx_r)$$

then we have natural associated operators

$$\begin{aligned} & \Lambda_{n,r}(f_n)(x_1, \ldots x_r) \\ = & \int f_n(x_1, \ldots, x_n) k_{n,r}(x_1, \ldots, x_r; dx_{r+1} \ldots dx_n) \end{aligned}$$

mapping functions of n variables to functions of r variables. E.g. the kernel for SIP is given by

$$k_{n,r}(x_1, \ldots, x_r; A) = p(\delta_{x_1} + \ldots \delta_{x_r})(A)$$

We can then define a general intertwiner

$$I_n(f_n, \eta) = \sum_{r=0}^{n} C_{n,r} J_r(\Lambda_{n,r} f_n, \eta) \tag{3}$$

where, as before,

$$J_r(g, \eta) = \int g(x_1, \ldots, x_r) \eta^{(r)}(dx_1 \ldots dx_r) \tag{4}$$

and $C_{n,r}$ are arbitrary constants. Then we have.

- Under consistency and compatibility we have the intertwining

$$\mathbb{E}_\eta I_n(f_n, \eta(t)) = I_n(p_t^{(n)} f_n, \eta) \tag{5}$$

- In the reversible case this leads to the self-duality

$$\frac{d\mathbb{E}_\eta I_n(\cdot, \eta(t))}{d\lambda^{\otimes n}}(y_1, \ldots, y_n) = I_n(p_t^{(n)}(y_1, \ldots, y_n; \cdot), \eta)$$

For appropriate choices of $C_{n,r}$ this leads to self- duality with products of Meixner polynomials for the generalized SIP.

Additional remarks, perspectives

1. Boundary driven processes in the continuum. For the independent case L. Bertini and G. Posta (2019) defined a process of Brownian motions on the interval coupled to boundary reservoirs. Jointly with G. Carinci, S. Floreani, C. Giardinà, we show factorial intertwining, duality as well as orthogonal duality.
2. We also show how this process can be obtained as a scaling limit of boundary driven random walks. Also the interval setting generalizes.
3. Asymmetric processes: here we need an appropriate modification of consistency.
4. Beyond Markov? Processes with memory and exclusion?