Integrability, supersymmetry and duality for vicious walkers with pair creation

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1. Definitions and Notation
2. Results: Self-Duality, duality and current distribution
3. Proofs

1 Definitions and Notation

1.1 State space

- Finite lattice Λ_L with L sites:
  (a) Torus $\mathbb{T}_L := \mathbb{Z}/L\mathbb{Z} = \{0, 1, 2, \ldots, L-1\}$ (periodic boundary conditions)
  (b) Finite box $\mathbb{B}_L := \{1, 2, \ldots, L\}$

- Local state space $\mathbb{S} := \{0, 1\}$ and occupation variables $\eta_k \in \mathbb{S}$ for $k \in \Lambda_L$

- Configuration $\boldsymbol{\eta} := \{\eta_0, \ldots, \eta_{L-1}\} \in \mathbb{S}^L$ or $\mathbf{k} := \{k : \eta_k = 1\}$.

- Single flip configuration $\bar{\boldsymbol{\eta}}^l$ with occupation variables

$$\bar{\eta}_k^l = \begin{cases} 1 - \eta_k & \text{if } k = l \\ \eta_k & \text{else.} \end{cases}$$

- Double flip configuration $\bar{\boldsymbol{\eta}}^{ll+1}$ with occupation variables

$$\bar{\eta}_k^{ll+1} = \begin{cases} 1 - \eta_k & \text{if } k \in \{l, l+1\} \\ \eta_k & \text{else.} \end{cases}$$

1.2 Deposition–annihilation process (DAP)

Independent vicious random walkers with deposition of particle pairs:

Process	Transition	Rate
Nearest neighbour particle jumps	$10 \leftrightarrow 01$	w
Nearest neighbour pair deposition	$00 \to 11$	μ
On-site instantaneous pair annihilation	$2 \to 0$	∞

Remark: DAP is equivalent to exclusion process with nearest-neighbour pair deposition and nearest-neighbour pair annihilation $11 \to 00$ with finite rate $\lambda = 2w + \mu$

- Local transition rate from $\boldsymbol{\eta} \to \bar{\boldsymbol{\eta}}^{kk+1}$: $w_k(\boldsymbol{\eta}) := \mu + w\left(\eta_k + \eta_{k+1}\right)$.
- Transition rate from $\boldsymbol{\eta} \to \boldsymbol{\eta}'$:

$$w_{\boldsymbol{\eta}'\boldsymbol{\eta}} = \sum_{k \in \Lambda} w_k(\boldsymbol{\eta}) \delta_{\boldsymbol{\eta}', \bar{\boldsymbol{\eta}}^{kk+1}}$$

- Particle number parity (even/odd) is conserved

Generator $\mathcal{L}$:

$$\text{Torus:} \quad \mathcal{L}f(\boldsymbol{\eta}) = \sum_{k=0}^{L-1} w_k(\boldsymbol{\eta})[f(\boldsymbol{\eta}^{k\,k+1}) - f(\boldsymbol{\eta})]$$

$$\text{Box:} \quad \mathcal{L}f(\boldsymbol{\eta}) = \sum_{k=1}^{L-1} w_k(\boldsymbol{\eta})[f(\boldsymbol{\eta}^{k\,k+1}) - f(\boldsymbol{\eta})]$$

- DAP is reversible w.r.t. the Bernoulli product measure $\pi^*(\boldsymbol{\eta}) = \prod_{k \in \Lambda_L} \pi^*(\eta_k)$
- Marginals: $\pi^*(\eta_k) = \dfrac{(1-\eta_k)\sqrt{\lambda} + \eta_k\sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}}$
- Particle density $\rho^* := \langle \eta_k \rangle_{\pi^*} = \dfrac{\sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}}. < \dfrac{1}{2}$
- Unique invariant measures of the even and odd sectors: $\pi^{*\pm}(\boldsymbol{\eta}) = \dfrac{1 \pm (-1)^{N(\boldsymbol{\eta})}}{1 \pm (1 - 2\rho^*)^L}\pi^*(\boldsymbol{\eta})$

1.3 Independent two-level atoms (EEP)

Excitation-emission process (EEP) $= M$ independent two-level atoms

Process	Transition	Rate
Excitation	$0 \to 1$	μ_r
Emission	$1 \to 0$	λ_r

Remark: For one atom the EEP is a generic two-state Markov process

- Single-atom transition rate from $\boldsymbol{\eta} \to \boldsymbol{\eta}^r$: $w_r(\eta_r) = \mu_r(1-\eta_r) + \lambda_r \eta_r$

Generator:

$$\mathcal{G}_r = \sum_{r \in \mathbb{B}_M} \mathcal{G}_r f(\boldsymbol{\eta}) = w_r(\boldsymbol{\eta})[f(\bar{\boldsymbol{\eta}}^r) - f(\boldsymbol{\eta})]$$

- EEP is reversible w.r.t. the Bernoulli product measure $\pi^{EEP}(\boldsymbol{\eta}) = \prod_{r \in \mathbb{B}_M} \pi_r^{EEP}(\eta_r)$

- Marginals: $\pi_r^{EEP}(\eta_r) = \dfrac{(1-\eta_r)\lambda_r + \eta_r \mu_r}{\lambda_r + \mu_r}, \qquad \rho_r^{EEP*} = \dfrac{\mu_r}{\lambda_r + \mu_r}$

1.4 Incidence matrix

Incidence matrix $I \equiv -H$: Matrix elements $H_{\boldsymbol{\eta}'\boldsymbol{\eta}} = \begin{cases} -w(\boldsymbol{\eta} \to \boldsymbol{\eta}') & \boldsymbol{\eta} \neq \boldsymbol{\eta}' \\ \sum'_{\boldsymbol{\eta}'} w(\boldsymbol{\eta} \to \boldsymbol{\eta}') & \boldsymbol{\eta} = \boldsymbol{\eta}'. \end{cases}$

- Eqivalent definitions via $\mathcal{L}f(\boldsymbol{\eta}) = -\sum_{\boldsymbol{\eta}' \in \mathbb{S}} f(\boldsymbol{\eta}')H_{\boldsymbol{\eta}'\boldsymbol{\eta}}$ with bijective mapping between state space $\mathbb{S}$ with cardinality d and vector space $\mathfrak{S} := \mathbb{C}^d$ with canonical basis vectors $\langle\, \boldsymbol{\eta} \,| \equiv \mathfrak{e}_{\imath(\boldsymbol{\eta})}$ (represented as row vectors)

- A probability measure $p(\boldsymbol{\eta})$ is represented by column vector $|\, p \,\rangle := \sum_{\boldsymbol{\eta}} p(\boldsymbol{\eta})|\, \boldsymbol{\eta} \,\rangle$ and the diagonal probability matrix $\hat{P}^* := \sum_{\boldsymbol{\eta}} p(\boldsymbol{\eta})|\, \boldsymbol{\eta} \,\rangle \otimes \langle\, \boldsymbol{\eta} \,|$

- Invariant measure: $H|\, \pi^* \,\rangle = 0$

- Reversed process: $H^{rev} = \hat{\pi}^* H^T (\hat{\pi}^*)^{-1}$

Remark: Lattice gas models with local state space $\mathbb{S} = \{0, 1, \ldots, d-1\}$: Lexigraphic ordering $\imath(\boldsymbol{\eta}) = 1 + \sum_{k=0}^{L-1} \eta_k d^{L-1-k}$ induces tensor basis of canonical basis vectors $\langle\, \boldsymbol{\eta} \,|$ of $\mathfrak{S}^L \cong (\mathbb{C}^d)^{\otimes L}$

$L = 1 : \langle\, 0 \,| = (1, 0, 0, \ldots), \langle\, 1 \,| = (0, 1, 0, \ldots),$

$L > 1 : \langle\, \boldsymbol{\eta} \,| = \langle\, \eta_0 \,| \otimes \langle\, \eta_1 \,| \otimes \cdots \otimes \langle\, \eta_{L-1} \,|$

1.5 Duality

Consider two Markov processes x_t and ω_t with countable state spaces X and Ω.

Definition 1.1 *The two processes x_t and ω_t are said to be dual to each other with respect to the duality function $D : X \times \Omega \mapsto \mathbb{R}$ if*

$$\langle\, D(x, \omega_t)\, \rangle_\omega = \langle\, D(x_t, \omega)\, \rangle_x \quad \forall t \geq 0$$

for initial values ω and x respectively.

Duality in matrix form [Sudbury and Lloyd (1995)]: Define the duality matrix

$$D = \sum_{x \in X} \sum_{\omega \in \Omega} D(x, \eta) |\, x\, \rangle \otimes \langle\, \omega\, |$$

$\Rightarrow$ Duality for processes with generators H and G :

$$DH = G^T D.$$

$\Rightarrow$ Use of duality: Exploit symmetries to express properties of one process in terms of another (possibly simpler) one.

Selfduality: Exploits symmetries of the generator [GMS and Sandow (1994) for SEP and SPEP, Giardinà et al. (2009) for general setting]

Theorem 1.2 *Let H be the negative incidence matrix of a Markov process η_t with countable state space and invariant measure π^* and H^{rev} be the negative incidence matrix of the reversed process. Assume that there exists a matrix S such that*

$$SH = H^{rev} S.$$

Then H is self-dual with duality function $D(\xi, \eta)$ given by the duality matrix

$$D = (\hat{\pi}^*)^{-1} S.$$

• The proof is elementary and follows from the chain of equalities
$DH = (\hat{P}^*)^{-1} SH = (\hat{P}^*)^{-1} H^{rev} S = (\hat{P}^*)^{-1} H^{rev} \hat{P}^* D = H^T D.$

• This is a slightly stronger version of Theorem 2.6 in Giardinà et al. (2009) by not making any assumption on the existence of S^{-1}.

• It follows that if H is reversible then the hypothesis the theorem reads $SH = HS$, i.e. S is a symmetry of H.

2 Duality results for the DAP

2.1 Results from supersymmetry, $\Lambda_L = \mathbb{B}_L$

Linear algebra conventions:

Definition 1.3 *(i) The Pauli matrices are the* 2×2 *matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Spin ladder operators and projectors:

$$\sigma^{\pm} := \frac{1}{2}\left(\sigma^1 \pm i\sigma^2\right), \quad \hat{n} := \frac{1}{2}\left(\mathbb{1} - \sigma^3\right), \quad \hat{v} := \frac{1}{2}\left(\mathbb{1} + \sigma^3\right).$$

Related notation: $\sigma^0 \equiv \mathbb{1}$ for the two-dimensional unit matrix and $\mathbf{1}$ for the generic unit matrix.

Definition 1.4 *(i) For any $L \geq 1$ and any two-dimensional matrix s the local operators s_k with $k \in \mathbb{T}_L$ are defined by the Kronecker product*

$$s_k := \mathbb{1}^{\otimes k} \otimes s \otimes \mathbb{1}^{\otimes (L-k-1)}$$

with the conventions $s^{\otimes 0} = \mathbb{1}$ and $s^{\otimes 1} = s$.

(ii) For any $L \geq 2$ and any four-dimensional matrix h, local operators $h_{k,k+1}$ with $k \in \mathbb{T}_L$ are defined by the Kronecker product

$$h_{k,k+1} := \mathbb{1}^{\otimes k} \otimes h \otimes \mathbb{1}^{\otimes (L-k-2)}.$$

The matrices $h_{L-1,L}$ are defined via the decomposition $h = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} f_{\alpha\beta} \sigma^{\alpha} \otimes \sigma^{\beta}$ by the Kronecker product

$$h_{L-1,L} = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} f_{\alpha\beta} \sigma^{\alpha} \otimes \mathbb{1}^{\otimes (L-2)} \otimes \sigma^{\beta}.$$

Note: For $k \in \mathbb{B}_L$ the definitions are similar, with the shift $k \to k-1$ in the tensor products.

Algebra conventions: For elements A, B of some associative algebra over $\mathbb{C}$ and $q \in \mathbb{C} \setminus \{0\}$ we denote by

$$[A, B] := AB - BA, \quad \{A, B\} := AB + BA, \quad [A]_q := \frac{q^A - q^{-A}}{q - q^{-1}}$$

the commutator, anticommutator, and q-symbol respectively.

Definition 1.5 *For $u, v \in \mathbb{C}$, the quantum superalgebra $U_{u,v}[\mathfrak{gl}(1|1)]$ is the associative algebra over $\mathbb{C}$ with generators T^0, T^1, T^2, T^3 subject to the following relations:*

$$\begin{aligned} &\left\{T^1, T^1\right\} = 2[T^0]_u, \quad \left\{T^2, T^2\right\} = 2[T^0]_v, \quad \left\{T^1, T^2\right\} = 0 \\ &[T^0, T^1] = [T^0, T^2] = [T^0, T^3] = 0 \\ &[T^3, T^1] = iT^2, \quad [T^3, T^2] = -iT^1. \end{aligned}$$

The subalgebra generated by T^0, T^1, T^2 is called the Hinrichsen-Rittenberg (HR) algebra.

Note: Replacing the anticommutators by commutators and taking $u = v =: q$ one obtains the defining relations of the q-deformed universal envelopping algebra $U_q[\mathfrak{gl}(2, \mathbb{C})]$.

Jordan-Wigner transformation ($1 \leq k \leq L$):

$$c_k^\alpha = \left(\prod_{l=1}^{k-1} \sigma_l^3 \right) \sigma_k^\alpha , \quad \alpha \in \{1, 2\}, \quad c_k = c_k^1 + i c_k^2 , \quad c_k^\dagger = c_k^1 - i c_k^2$$

$\Rightarrow$ Fermionic anticommutation relations $\{c_k , c_l\} = 0$, $\left\{ c_k^\dagger , c_l \right\} = \delta_{k,l}$

Matrix representation of HR algebra for $u, v \in \mathbb{C} \setminus \{0\}$: [Hinrichsen and Rittenberg (1992)]

$$T_u^1 = u^{-(L+1)/2} \sum_{k=1}^{L} u^k c_k^1 , \quad T_v^2 = v^{-(L+1)/2} \sum_{k=1}^{L} v^k c_k^2 , \quad T^0 = L\mathbf{1}$$

Corollary 1.6 *For any $a \in \mathbb{C} \setminus \{0\}$ and any invertible matrix $\hat{\pi}^*$ the matrices*

$$C^0 := T^0 , \quad C^1 := (\hat{\pi}^*)^{\frac{1}{2}} T_a^1 (\hat{\pi}^*)^{-\frac{1}{2}} , \quad C^2 := (\hat{\pi}^*)^{\frac{1}{2}} T_{a^{-1}}^2 (\hat{\pi}^*)^{-\frac{1}{2}}$$

form a representation of the HR algebra with $u = a$ and $v = a^{-1}$.

Remark: The matrices C^1 and C^2 change a state vector with N particles into a linear combination of state vectors with $N \pm 1$ particles, i.e., they change the particle number parity.

Theorem 1.7 *For any $\delta_i \in \mathbb{R}$, $i \in \{1,2,3,4\}$ and $a = 2\rho^* - 1$ the DAP defined on $\mathbb{B}_L$ with jump rate $w = 1$ is selfdual w.r.t. the duality matrix*

$$D = \hat{\pi}^{-1} \left(\delta_0 \mathbf{1} + \delta_1 C^1 + i\delta_2 C^2 + i\delta_3 C^1 C^2 \right)$$

which is the most general duality matrix arising from the supersymmetry alone.

Corollary 1.8 *The even and odd sectors of the DAP are dual to each other w.r.t. the duality matrix D with $\delta_0 = \delta_3 = 0$.*

Remarks:

(1) The DAP defined on $\mathbb{B}_L$ is an integrable model which implies further symmetries and hence allows for more general duality functions in conjunction with the supersymmetry.

(2) The supersymmetry is broken on the torus $\mathbb{T}_L$ but integrability is preserved.

2.2 Results from integrability, $\Lambda_L = \mathbb{T}_L$

- Some short-hand notation: $\phi^{\pm} = (1 \pm 1)/2, \quad \mathrm{t}_r^{\pm} := \tan\left(\frac{\pi(2r+\phi^{\pm})}{2L}\right)$

$\mathrm{c}_r^{\pm} := \cos\left(\frac{\pi(2r+\phi^{\pm})}{L}\right), \quad \mathrm{s}_r^{\pm} := \sin\left(\frac{\pi(2r+\phi^{\pm})}{L}\right),$

- Two specific choices of EEPs with generators $\mathcal{G}^{\pm}$:

Fix an integer $L \in \mathbb{N}$. Then the EEPs with generators $\mathcal{G}^{\pm}$ are defined by the index sets

$$\hat{\mathbb{I}}^{+} := \{0, \ldots, M^{+} - 1\} \text{ with } M^{+} = [L/2]$$
$$\hat{\mathbb{I}}^{-} := \{1, \ldots, M^{-}\} \text{ with } M^{-} = [(L-1)/2]$$

and the rates

$$\mu_r^{\pm} = 2\mu\left(1 + \mathrm{c}_r^{\pm}\right), \quad \lambda_r^{\pm} = 2\lambda\left(1 - \mathrm{c}_r^{\pm}\right)$$

• Discrete Fourier transform of Jordan-Wigner operators:

$$b_r^{\pm} := \frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\sqrt{L}} \sum_{k=0}^{L-1} \mathrm{e}^{i\frac{\pi k(2r+\phi^{\pm})}{L}} c_k .$$

• Cooper pair operators: $B_r^+ := b_r^+ b_{-r-1}^+, \quad B_r^- := b_r^- b_{-r}^-$.

$\Rightarrow$ Subspaces $\mathfrak{V}^{\pm}$ of $\mathbb{C}^{2^L}$ with reference states $|\,0^+\,\rangle$ (empty lattice), $|\,0^-\,\rangle$ (uniformly distributed single particle) and $2N$-particle states $B_{r_1}^{+\dagger} \ldots B_{r_N}^{+\dagger} |\,0^+\,\rangle$, $(2N+1)$-particle states $B_{r_1}^{-\dagger} \ldots B_{r_N}^{-\dagger} |\,0^-\,\rangle$

Definition 1.9 *Let $\mathbb{I}$ be a finite set of cardinality N and S^1 be the unit circle in the complex plane. For integers k_i, p_i, $i \in \mathbb{I}$ and $z_{k_i} \in S^1$ the determinant*

$$S_{\mathbf{k}}(p_1, \ldots, p_N) = \det \begin{vmatrix} z_{k_1}^{p_1} & z_{k_1}^{p_2} & \ldots & z_{k_1}^{p_N} \\ z_{k_2}^{p_1} & z_{k_2}^{p_2} & \ldots & z_{k_2}^{p_N} \\ \ldots & \ldots & \ldots & \ldots \\ z_{k_N}^{p_1} & z_{k_N}^{p_2} & \ldots & z_{k_N}^{p_N} \end{vmatrix}$$

is called the plane-wave Slater determinant.

Theorem 1.10 *Let* $\mathbf{k}$ *be a configuration of the DAP with* N *particles at positions* k_i *and* $\mathbf{r}$ *be a configuration of an EEP with* N *excited atoms* r_i. *The projected DAP with the sector generator* $\mathcal{H}^{\pm}$ *and the EEP with generator* $\mathcal{G}^{\pm}$ *are dual with respect to the determinantal duality functions*

$$D^{+}(\mathbf{r},\mathbf{k}) = \frac{1}{(iL)^{N}} S_{\mathbf{k}}(r_1+\frac{1}{2},-r_1-\frac{1}{2},\ldots,r_N+\frac{1}{2},-r_N-\frac{1}{2})$$

$$D^{-}(\mathbf{r},\mathbf{k}) = \frac{1}{(iL)^{N+\frac{1}{2}}} S_{\mathbf{k}}(0,r_1,-r_1,\ldots,r_N,-r_N).$$

Remarks:

1) In terms of occupation variables $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ the duality function has the matrix product form

$$D^{\pm}(\boldsymbol{\xi},\boldsymbol{\eta}) = \langle\, \boldsymbol{\eta}\,|\prod_{r\in\mathbb{I}^{\pm}}\left(\frac{\lambda}{\mu}\mathrm{t}_r^{\pm}B_r^{\pm\dagger}\right)^{\xi_r}|\,0^{\pm}\,\rangle$$

with the reference states $|\,0^{\pm}\,\rangle$ and the pair-creation matrices $B_r^{\pm\dagger}$.

2) Formal analogies: (i) Matrix product form of the duality function vs. factorized duality functions [Redig and Sau, 2018] , (ii) invariant matrix product measures [Derrida et al. 1993] vs. conventional invariant product measures, (iii) matrix product states [Klümper et al., 1991] vs. pure states in quantum systems

3) Also the Slater determinant has its origin in the study of many-body quantum systems

Current distribution in the DAP:

Definition 1.11 *For the DAP let $J_k^+(t)$ be the number of particle jumps in clockwise (positive) direction across bond $(k, k+1)$ up to time t and $J_k^-(t)$ be the number of particle jumps in anticlockwise (negative) direction up to time t. The random variable*

$$J(t) := \sum_{k=0}^{L-1} [J_k^+(t) - J_k^-(t)]$$

is called the integrated current and

$$j(t) := \frac{1}{\sqrt{L}\sqrt{t}} \sum_{k=0}^{L-1} [J_k^+(t) - J_k^-(t)]$$

is called the rescaled integrated current.

- Generating functions: $F_L(s;t) := \langle\, e^{sJ(t)} \,\rangle$, $f_L(s;t) := \langle\, e^{sj(t)} \,\rangle = F_L(\frac{s}{\sqrt{Lt}};t)$
- Cumulant generating functions: $C_L(s;t) := \ln F_L(s;t)$, $c_L(s;t) := \ln f_L(s;t)$

Theorem 1.12 *Let the DAP$^{\pm}$ start at time $t = 0$ with an initial measure that is a projected Bernoulli measure with arbitrary density $\rho_0 \in [0, 1]$. With $f(s) := 2w(1 - \cosh s)$ the asymptotic rescaled cumulant function*

$$C(s) := \lim_{L\to\infty} \frac{1}{L} \lim_{t\to\infty} \frac{1}{t} C_L^{\pm}(s; t)$$

of the integrated current $J(t)$ is given by

$$C(s) = \int_0^{\pi} \frac{\mathrm{d}x}{2\pi} \left[\sqrt{[\mu + \lambda + (\mu - \lambda + f(s)) \cos x]^2 - 4f(s)\mu\,(1 + \cos x)} - (\mu + \lambda) \right],$$

independently of the initial density ρ_0 and the sector. Moreover, the asymptotic distribution of the rescaled current $j(t) = J(t)/\sqrt{Lt}$ is normal with variance

$$\sigma^2 = \mu \frac{1 - 2\rho^*}{\rho^*}$$

where $\rho^* \in [0, 1/2)$ is the stationary density of the process.

Remarks:

(1) Correlations lead to a non-normal distribution of the random variable $J(t)$ for large L and t. However, the deviations from normal behaviour are smaller than of order $\sqrt{Lt}$ as the result for the rescaled current shows.

(2) The vanishing of the variance at $\rho^* = 1/2$ is trivial since this density is stationary for the limiting case $w = 0$ when no jumps occur. The variance vanishes also at $\rho^* = 0$ corresponding to $\mu = 0$ where the system reaches an absorbing state with no current (empty lattice in the even sector) or ends up with single particle (odd sector) with vanishing rescaled current.

(3) Fluctuations of other quantities such as the right- or left currents $J^{\pm}(t)$ or the particle number $N(t)$ at time t can be obtained in a similar fashion for arbitrary initial densities ρ_0 using the duality with the EEP.

3 Proofs

Preliminaries

Incidence matrix for DAP: Define diagonal matrix $\hat{w}_k := \mu\mathbf{1} + w(\hat{n}_k + \hat{n}_{k+1})$. Then

$$H = \sum_{k\in\Lambda_L} h_{k\,k+1} = \sum_{k\in\Lambda_L} (\mathbf{1} - \sigma_k^1\sigma_{k+1}^1)\hat{w}_k$$

Expression of local generator in terms of spin ladder operators:

$$h_{k\,k+1} = \mu\mathbf{1} + w(\hat{n}_k + \hat{n}_{k+1}) - w(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+) - \mu\sigma_k^-\sigma_{k+1}^- - \lambda\sigma_k^+\sigma_{k+1}^+$$

Expression in terms of Jordan-Wigner operators:

$$\sigma_k^+\sigma_{k+1}^- = c_{k+1}^\dagger c_k, \quad \sigma_k^-\sigma_{k+1}^+ = c_k^\dagger c_{k+1}$$

$$c_{k+1}c_k = \sigma_k^+\sigma_{k+1}^+, \quad c_k^\dagger c_{k+1}^\dagger = \sigma_k^-\sigma_{k+1}^-$$

$$\hat{n}_k = c_k^\dagger c_k$$

Even and odd sectors: Define number operator and projectors as

$$\hat{N} := \sum_{k \in \Lambda_L} \hat{n}_k, \quad \hat{P}^{\pm} := \frac{1}{2}\left(1 \pm (-1)^{\hat{N}}\right)$$

$\Rightarrow$ Incidence matrix for each sector:

$$H^{\pm} = H\hat{P}^{\pm}$$

• Invariant measures in vector form:

Bernoulli product measure: $|\,\pi^*\,\rangle = \begin{pmatrix} 1-\rho^* \\ \rho^* \end{pmatrix}^{\otimes L}$

Projected Bernoullis: $|\,\pi^{*\pm}\,\rangle = \dfrac{1 \pm (-1)^{\hat{N}(\boldsymbol{\eta})}}{1 \pm (1-2\rho^*)^L}|\,\pi^*\,\rangle.$

Incidence matrix of EEP: Define the diagonal matrix $\hat{w}_r^{EEP} = \mu_r \hat{v}_r + \lambda_r \hat{n}_r$. Then

$$G = \sum_{r \in \mathbb{I}} g_r \text{ with } g_r = (\mathbf{1} - \sigma_r^1)\hat{w}_r^{EEP}$$

Expression of single generator in terms of spin ladder operators:

$$g_r = \mu_r(\hat{v}_r - \sigma_r^-) + \lambda_r(\hat{n}_r - \sigma_r^+)$$

Commutation relations:

$$[\sigma_k^+, \sigma_l^-] = \delta_{k,l}(\mathbf{1} - 2\hat{n}_k), \quad [\sigma_k^\pm, \hat{n}_l] = \pm\delta_{k,l}\sigma_k^\pm$$

- Pauli matrices commute at different sites (bosons) and satisfy $\mathfrak{su}(2)$ commutation relations at equal sites
- Spin-1/2 representation $\leftrightarrow$ absence of double occupancy $\leftrightarrow$ hard-core bosons

3.1 Proof of selfduality and parity duality

• Ground state transformation to the quantum Hamiltonian $H^{XY} = (\hat{\pi}^*)^{-\frac{1}{2}} H (\hat{\pi}^*)^{\frac{1}{2}}$ of the free-fermion quantum XY chain (straightforward computation):

$$h^{XY}_{kk+1} = -\frac{w}{2}\left[a\sigma^1_k\sigma^1_{k+1} + a^{-1}\sigma^2_k\sigma^2_{k+1} - \sigma^3_k - \sigma^3_{k+1}\right] + (w+\mu)\mathbf{1}, \quad a = 1 - 2\rho^*$$

• Jordan-Wigner transformation: H^{XY} commutes with the generators $T^{0,1,2}$ of the HR algebra [Hinrichsen and Rittenberg (1992)].

$\Rightarrow S = \delta_0 \mathbf{1} + \delta_1 C^1 + i\delta_2 C^2 + i\delta_3 C^1 C^2$ is a symmetry operator of the DAP.

• $\Rightarrow$ Selfduality w.r.t. D (duality-symmetry theorem)

• Uniqueness of D (constructed only from the supersymmetry): Anticommutation relations of the HR algebra imply $(T^1)^2 = (T^2)^2 = 0 \Rightarrow$ universal envelopping algebra of the HR algebra consists only of the four elements T^0, T^1, T^2, T^1T^2.

• Parity duality follows from the shift of particle parity induced by the generators T^1, T^2 which implies the anticommutation relations $\left\{T^{1,2}, (-1)^{\hat{N}}\right\} = 0$ and therefore $(\delta_1 C^1 + i\delta_2 C^2)P^{\pm} = P^{\mp}(\delta_1 C^1 + i\delta_2 C^2)$ for the projectors on the even and odd sectors.

3.2 Proof of duality with EEP

- Main idea: Work within $\mathfrak{V}^{\pm}$ and use algebra of the Cooper pair operators

Lemma 2.1 *Let $b_r^{\pm}$ and $b_r^{\pm\dagger}$ be the Fourier components of the free-fermion operators. The Cooper pair operators $B_r^{\pm\dagger}$, $B_r^{\pm}$ and the pair number operators*

$$\hat{N}_r^{+} := \frac{1}{2}\left(b_r^{+\dagger}b_r^{+} + b_{-r-1}^{+\dagger}b_{-r-1}^{+}\right), \quad \hat{N}_r^{-} := \frac{1}{2}\left(b_r^{-\dagger}b_r^{-} + b_{-r}^{-\dagger}b_{-r}^{-}\right),$$
$$\hat{I}_r^{+} := \frac{1}{2}\left(b_r^{+\dagger}b_r^{+} - b_{-r-1}^{+\dagger}b_{-r-1}^{+}\right), \quad \hat{I}_r^{-} := \frac{1}{2}\left(b_r^{-\dagger}b_r^{-} - b_{-r}^{-\dagger}b_{-r}^{-}\right),$$

all commute for mutually different indices. Moreover, $B_0^{-} = \hat{I}_0^{-} = 0$ for any L, $B_{\frac{L-1}{2}}^{+} = \hat{I}_{\frac{L-1}{2}}^{+} = 0$ for L odd, $B_{\frac{L}{2}}^{-} = \hat{I}_{\frac{L}{2}}^{-} = 0$ for L even.

Lemma 2.2 *Excluding $r = 0$ for any L, $r = \frac{L-1}{2}$ for L odd, and $r = \frac{L}{2}$ for L even, the following holds for all other r:*
(i) The matrices

$$\hat{L}_r^{\pm} := \mathbf{1} - 2\hat{N}_r^{\pm}$$

and the pair annihilation and creation operators $B_r^{\pm}$, $B_s^{\pm\dagger}$ satisfy the bosonic $\mathfrak{su}(2)$- commutation relations

$$[B_r^{\pm}, B_s^{\pm\dagger}] = \hat{L}_r^{\pm}\delta_{r,s} \tag{1}$$

$$[\hat{L}_r^{\pm}, B_s^{\pm}] = 2B_r^{\pm}\delta_{r,s} \tag{2}$$

$$[\hat{L}_r^{\pm}, B_s^{\pm\dagger}] = -2B_r^{\pm\dagger}\delta_{r,s}. \tag{3}$$

(ii) The matrix algebra (1) - (3) has central elements $\mathbf{1}$ and $\hat{I}_r^{\pm}$.

(iii) The local Casimir operator of $\mathfrak{sl}(2,\mathbb{C})$

$$C_r^{\pm} \quad := \quad B_r^{\pm}B_r^{\pm\dagger} + B_r^{\pm\dagger}B_r^{\pm} + \frac{1}{2}\left(\hat{L}_r^{\pm}\right)^2$$

is given by the linear combination $C_r^{\pm} = \dfrac{3}{2}\mathbf{1} - 6\left(\hat{I}_r^{\pm}\right)^2$ of the central elements.

Proposition 2.3 *With the pair vacancy operators $\hat{V}_r^{\pm} := \mathbf{1} - \hat{N}_r^{\pm}$ and the linear combinations*

$$
\begin{aligned}
h_0^- &:= 2\mu \hat{V}_0^-, \quad h_{\left[\frac{L}{2}\right]}^{\pm} := 2\lambda \hat{N}_{\left[\frac{L}{2}\right]}^{\pm} \\
h_r^{\pm} &:= \mu_r^{\pm}\left(\hat{V}_r^{\pm} - \mathrm{t}_r^{\pm} B_r^{\pm\dagger}\right) + \lambda_r^{\pm}\left(\hat{N}_r^{\pm} - (\mathrm{t}_r^{\pm})^{-1} B_r^{\pm}\right)
\end{aligned}
$$

the generator of the DAP takes the form

$$
H = \begin{cases}
\sum_{r=0}^{\left[\frac{L}{2}\right]-1} \hat{P}^+ h_r^+ + \hat{P}^- h_0^- + \sum_{r=1}^{\left[\frac{L-1}{2}\right]} \hat{P}^- h_r^- + \hat{P}^- h_{\left[\frac{L}{2}\right]}^- & L \text{ even} \\
\\
\sum_{r=0}^{\left[\frac{L}{2}\right]-1} \hat{P}^+ h_r^+ + \hat{P}^+ h_{\left[\frac{L}{2}\right]}^+ + \hat{P}^- h_0^- + \sum_{r=1}^{\left[\frac{L-1}{2}\right]} \hat{P}^- h_r^- & L \text{ odd}\,.
\end{cases}
$$

Remark: Notice the appearance of the EEP transition rates μ_r, λ_r

Proof: : Use JW representation of the generator $\rightarrow$ Lengthy but straightforward computations using the fermionic anticommutation relations of the JW operators.

Corollary 2.4 *The matrices h_r satisfy the commutativity property*

$$[h_r^{\pm}, H^{\pm}] = 0$$

for any $r \in \{0, \dots, \left[\frac{L}{2}\right]\}$.

Remarks:

(1) The corollary is a direct consequence of the bosonic commutation relations of the Cooper pairs established in Lemma 2.2 and reflects the integrability of the model, i.e., the existence of mutually commuting conserved quantities.

(2) The conserved quantities $h_r^{\pm}$ are not local on $\mathbb{T}_L$

Proof of Theorem 1.10 (Duality DAP-EEP):

(i) Transformation $\hat{V}_r^{\pm} := \left(\frac{\lambda}{\mu}\mathrm{t}_r^{\pm}\right)^{\hat{N}_r^{\pm}}$, $\hat{V}^{+} := \prod_{r=0}^{\left[\frac{L}{2}\right]-1} \hat{V}_r^{+}$, $\hat{V}^{-} := \prod_{r=1}^{\left[\frac{L-1}{2}\right]} \hat{V}_r^{-}$

$\tilde{g}_r^{\pm} := P^{\pm}\left(\hat{V}_r^{\pm}\right)^{-1}\left(h_r^{\pm}\right)^{T}\hat{V}_r^{\pm} = \mu_r\left(\hat{V}_r^{\pm} - B_r^{\pm\dagger}\right) + \lambda_r\left(\hat{N}_r^{\pm} - B_r^{\pm}\right)$

(ii) $\Rightarrow$ For $\tilde{G}^{\pm} = \sum_r \tilde{g}_r^{\pm}$:

$$\hat{V}^{\pm}H^{\pm} = \left(\tilde{G}^{\pm}\right)^{T}\hat{V}^{\pm}$$

(iii) Casimir $\hat{C}_r^{\pm} = 3/2$ on subspace $\mathfrak{V}^{\pm}$ spanned by Cooper pair creation operators $\Rightarrow$ spin-1/2 representation of $\mathfrak{su}(2)$

$\Rightarrow B_r^{\pm}, B_r^{\pm\dagger}, \hat{N}_r$ act like Pauli matrices on this subspace

$\Rightarrow \tilde{G}^{\pm}$ is generator of EEP (embedding of $G^{\pm}$ in $\mathbb{C}^{2^L}$), $\hat{V}^{\pm}$ is duality matrix

(iv) Matrix product form of duality function follows from (i) and Fock space structure of N-particle states within $\mathfrak{V}^{\pm}$

(v) Slater determinant representation due to fermionic anticommutation relations and orthogonality and normalization of the Fourier modes $\square$

Proof of Theorem 1.12 (Current distribution):

(i) Definition of integrated current $\Rightarrow \quad C_L^{\pm}(s;t) = \langle\, e^{-H(s)t}\, \rangle_{\rho_0}$ with tilted generator $H(s) = \sum_k h_{kk+1}(s)$ where

$$h_{kk+1}(s) = h_{kk+1} - w(e^{s}-1)\sigma_k^{+}\sigma_{k+1}^{-} - w(e^{-s}-1)\sigma_k^{-}\sigma_{k+1}^{+}$$

(ii) Projected Bernoulli initial distribution $|\,\rho^{\pm}\,\rangle \in \mathfrak{V}^{\pm}$ [GMS 1995]

(iii) $H(s)$ represented by $B_r^{\pm}, B_r^{\pm\dagger}, \hat{N}_r$

$\Rightarrow$ Action of $e^{-H(s)t}$ stays within $\mathfrak{V}^{\pm}$

$\Rightarrow$ Computation of $e^{-H(s)t}$ = exponentiation of a two-dimensional matrix

(iv) Limit $t \to \infty \Rightarrow$ only lowest eigenvalue of $H(s)$ in $\mathfrak{V}^{\pm}$ relevant, no dependence on initial distribution

Use Euler-Maclaurin summation formula to compute limit $L \to \infty$ $\square$

Remark: Finite-size and finite-time corrections readily accessible