Markov process representation of semigroups whose generators include negative rates

Florian Völlering

University of Leipzig

June 7th 2021

Duality and Motivation

Definition

Duality between Markov processes with duality function H:

$$\mathbb{E}_x H(X_t, y) = \mathbb{E}_y H(x, Y_t) \tag{1}$$

Duality of Markov semi-groups:

$$[S_t H(\cdot, y)](x) = [S_t^{dual} H(x, \cdot)](y) \tag{2}$$

Duality of Markov generators:

$$[LH(\cdot, y)](x) = [L^{dual} H(x, \cdot)](y) \tag{3}$$

Algebraic methods for obtaining duality do not guarantee that L^{dual} is a Markov generator. If it is not, then

$$\mathbb{E}_x H(X_t, y) = [S_t^{dual} H(x, \cdot)](y)$$

still holds (with $S_t^{dual} = e^{tL^{dual}}$), but the stochastic representation of S_t^{dual} is lost.

Basics

Notation

- E countable state space;
- $A = (r(x,y))_{x,y\in E}$ matrix on E.

A is a Markov generator if

(a) $r(x,y) \geq 0$ for all $x, y \in E$, $x \neq y$ (jump rates from x to y are non-negative);

(b) $r(x,x) = -\sum_{y\neq x} r(x,y)$ (exit rate from x matches entry rate from x to all possible y);

(c) $\| A \|_\infty = \sup_x \sum_y |r(x,y)| < \infty$ (for convenience, prevents explosion)

Basic linear algebra

$S_t := e^{tA}$ is well-defined if (c) is satisfied, (a) and (b) are only needed for the stochastic interpretation of the semi-group.

Potential term and Feynman-Kac formula

Problem 1

(b) is violated, $r(x,x) \neq -\sum_{y \neq x} r(x,y)$ for some x.

Solution: split off a potential term V

Write $A = \widehat{A} + V$, with matrix $\widehat{A} = (\widehat{r}(x,y))$ and potential $V = (V(x))$ given by

- $\widehat{r}(x,y) = r(x,y)$, $x \neq y$;
- $\widehat{r}(x,x) = -\sum_{y \neq x} \widehat{r}(x,y)$;
- $V(x) = r(x,x) - \widehat{r}(x,x)$.

$\Rightarrow$ $\widehat{A}$ is a Markov generator

Feynman-Kac formula for the semi-group

- $(\widehat{X}_t)$ is the Markov process generated by $\widehat{A}$;
- $S_t f(x) = \mathbb{E}_x \left(f(\widehat{X}_t) e^{\int_0^t V(\widehat{X}_s) ds} \right)$.

$\Rightarrow$ stochastic representation of semi-group S_t

Negative jump rates

Problem 2

(a) is violated, $r(x, y) < 0$ for some $x \neq y$. (But (b) is satisfied.)

We cannot simply split off a term

- $r_+(x, y) = \max(r(x, y), 0)$, $x \neq y$
- $r_-(x, y) = \max(-r(x, y), 0)$, $x \neq y$

$$
\begin{aligned}
Af(x) &= \sum_y r(x, y)[f(y) - f(x)] \\
&= \sum_y r_+(x, y)[f(y) - f(x)] - \sum_y r_-(x, y)[f(y) - f(x)]
\end{aligned}
$$

Negative jump rates

Problem 2

(a) is violated, $r(x, y) < 0$ for some $x \neq y$. (But (b) is satisfied.)

We cannot simply split off a term

- $r_+(x, y) = \max(r(x, y), 0)$, $x \neq y$
- $r_-(x, y) = \max(-r(x, y), 0)$, $x \neq y$

$$\begin{aligned} Af(x) &= \sum_y r(x, y)[f(y) - f(x)] \\ &= \sum_y r_+(x, y)[f(y) - f(x)] - \sum_y r_-(x, y)[f(y) - f(x)] \\ &= \sum_y r_+(x, y)[f(y) - f(x)] + \sum_y r_-(x, y)[(-f)(y) - (-f)(x)] \end{aligned}$$

$\Rightarrow$ We would like to have a way for f to change sign

A signed particle

Extended state space

- $\widehat{E} := E \times \{-1, +1\}$; (position and sign)
- $\widehat{f}(x, u) := u \cdot f(x)$ (extend f by multiplying by the sign)

Regular jumps

Define a Markov generator $\widehat{A}_+ = (\widehat{r}_+((x, u), (y, v)))$:

- $\widehat{r}_+((x, u), (y, v)) = r_+(x, y)\delta_u(v)$;
- $\widehat{A}_+\widehat{f}(x, u) = \sum_y r_+(x, y)[\widehat{f}(y, u) - \widehat{f}(x, u)]$

$\Rightarrow$ jumps according to the positive rates r_+, irrespective of the sign.

Sign-changing jumps

Define a Markov generator $\widehat{A}_- = (\widehat{r}_-((x, u), (y, v)))$:

- $\widehat{r}_-((x, u), (y, v)) = r_-(x, y)\delta_u(-v)$;
- $\widehat{A}_-\widehat{f}(x, u) = \sum_{y \neq x} r_-(x, y)[\widehat{f}(y, -u) - \widehat{f}(x, u)]$

$\Rightarrow$ r_--jumps change the position like r_+-jumps, but include a sign flip.

A signed particle

The signed particle

- $\widehat{A} = \widehat{A}_+ + \widehat{A}_-$ is the Markovian generator;
- $(\widehat{X}_t, \widehat{U}_t)$ is the Markov process generated by $\widehat{A}$.

Comparison with the original generator A

$$Af(x) = \sum_y r_+(x,y)[f(y) - f(x)] + \sum_y r_-(x,y)[(-f)(y) - (-f)(x)]$$

$$= \underbrace{\widehat{A}_+\widehat{f}(x,+1) + \widehat{A}_-\widehat{f}(x,+1)}_{= \widehat{A}\widehat{f}(x,+1)} + \underbrace{\sum_{y \neq x}(r_-(x,y) + r_-(x,y))\widehat{f}(x,+1)}_{=: \widehat{V}(x,+1)\widehat{f}(x,+1)}$$

Compensating potential

- $\widehat{V}(x,u) := \sum_{y \neq x} 2r_-(x,y)$ is needed.

Stochastic representation of the semi-group

Theorem 1 (V. 2020)

Assume

- $A = (r(x,y))$ *a generator which may have negative off-diagonal entries;*
- $r(x,x) = -\sum_{y \neq x} r(x,y)$;
- $S_t = e^{tA}$ *semi-group generated by* A.

Then the semi-group has the stochastic representation

$$S_t f(x) = \mathbb{E}_{x,+1}\left(\widehat{U}_t f(\widehat{X}_t) e^{\int_0^t \widehat{V}(\widehat{X}_s) ds}\right),$$

where

- $(\widehat{X}_t, \widehat{U}_t)$ *is the signed particle started in* $(x, +1)$;
- $\widehat{V}(x) = \widehat{V}(x, \pm 1) = \sum_{y \neq x} 2r_-(x,y)$ *is the compensating potential.*

Proof

- $\mathbb{E}_{x,+1}\left(\widehat{U}_t f(\widehat{X}_t) e^{\int_0^t \widehat{V}(\widehat{X}_s)ds}\right)$ is the Feynman-Kac formulation of the solution of

$$\begin{cases} \frac{\partial \phi_t}{\partial t}(x,u) = \left(\widehat{A} + \widehat{V}(x,u)\right)\phi_t(x,u), \\ \phi_0 = \hat{f}. \end{cases}$$

- By the compensation formula, for any $\widehat{g}(x,u) = u \cdot g(x)$,

$$u \cdot Ag(x) = \left(\widehat{A} + \widehat{V}(x,u)\right)\widehat{g}(x,u).$$

- $\tilde{\phi}_t(x,u) := u \cdot S_t f(x)$ also satisfies

$$\begin{cases} \frac{\partial \tilde{\phi}_t}{\partial t}(x,u) = u \cdot AS_t f(x) = \left(\widehat{A} + \widehat{V}(x,u)\right)\tilde{\phi}_t(x,u), \\ \tilde{\phi}_0 = \hat{f}. \end{cases}$$

$\Rightarrow$ the solutions are equal.

A stochastic representation for arbitrary semi-groups

Theorem 2

Assume

- $A = (r(x,y))$ *is a matrix with* $\| A \|_\infty < \infty$;
- $S_t = e^{tA}$ *semi-group generated by* A.

Then the semi-group has the stochastic representation

$$S_t f(x) = \mathbb{E}_{x,+1}\left(\widehat{U}_t f(\widehat{X}_t) e^{\int_0^t \widehat{V}(\widehat{X}_s)ds}\right),$$

where

- $(\widehat{X}_t, \widehat{U}_t)$ *is the signed particle started in* $(x,+1)$ *as before;*
- $\widehat{V}(x) = \widehat{V}(x, \pm 1) = \sum_{y \neq x} | r(x,y) | - r(x,x)$ *is the potential.*

Example: Double Laplacian

Laplacian and Double Laplacian on $\mathbb{Z}$

$$\Delta f(x) = \frac{1}{2}(f(x+1) - f(x)) + \frac{1}{2}(f(x-1) - f(x))$$

$$\Delta\Delta f(x) = \frac{1}{4}(f(x+2) - f(x)) + \frac{1}{4}(f(x-2) - f(x)) - (f(x+1) - f(x)) - (f(x-1) - f(x)),$$

The signed particle

- $(x,u) \to (x \pm 2, u)$ at rate $\frac{1}{4}$;
- $(x,u) \to (x \pm 1, -u)$ at rate 1.

Number of sign flips

- $\sum_y r_-(x,y) = 2 \quad$ (unform rate of flipping sign);
- number N_t of sign changes is Poisson($2t$);
- N_t is even iff $\widehat{X}_t - \widehat{X}_0$ is even.

Stochastic representation of the double Laplacian semi-group

The different parts of the representation

- $e^{\int_0^t \widehat{V}(\widehat{X}_s)ds} = e^{4t}$;
- $\mathbb{P}(\widehat{X}_t - \widehat{X}_0 \text{ is even}) = \frac{1}{2}(1 + e^{-4t})$;

The general case

$$\begin{aligned} S_t f(x) &= e^{4t}\mathbb{E}_{x,+1}\left(\widehat{U}_t f(\widehat{X}_t)\right) \\ &= \frac{1}{2}(e^{4t}+1)\mathbb{E}_x\left[f(\widehat{X}_t)\middle|\widehat{X}_t - x \text{ even}\right] \\ &\quad - \frac{1}{2}(e^{4t}-1)\mathbb{E}_x\left[f(\widehat{X}_t)\middle|\widehat{X}_t - x \text{ odd}\right] \end{aligned}$$

If f is restricted to even sites

$$S_t f(x) = \begin{cases} \frac{1}{2}(e^{4t}+1)\mathbb{E}_x\left[f(\widehat{X}_t)\middle|\widehat{X}_t \text{ even}\right], & x \text{ even}; \\ -\frac{1}{2}(e^{4t}-1)\mathbb{E}_x\left[f(\widehat{X}_t)\middle|\widehat{X}_t \text{ odd}\right], & x \text{ odd}. \end{cases}$$

Example: all rates negative

The setting

- $r(x, y) \leq 0$ for all $x \neq y$;
- $\sum_{y \neq x} r_-(x, y) = \lambda_1$ for all x;
- $r(x, x) = \lambda_2$ for all x.

First observations

- $\widehat{V}(x, u) = 2\lambda_1 + \lambda_2 \Rightarrow e^{\int_0^t \widehat{V}(\widehat{X}_s) ds} = e^{(2\lambda_1 + \lambda_2)t}$;
- $\widehat{U}_t = (-1)^{N_t}$ with N_t the number of jumps;
- $\mathbb{P}(N_t \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda_1 t})$.

Stochastic representation of the semi-group

$$e^{At} f(x) = e^{(2\lambda_1 + \lambda_2)t} \mathbb{E}_{x,+1} \left[f(\widehat{X}_t) \;\middle|\; N_t \text{ even} \right] \frac{1}{2}(1 + e^{-2\lambda_1 t})$$
$$- e^{(2\lambda_1 + \lambda_2)t} \mathbb{E}_{x,+1} \left[f(\widehat{X}_t) \;\middle|\; N_t \text{ odd} \right] \frac{1}{2}(1 - e^{-2\lambda_1 t})$$

Example: all rates negative

Stochastic representation of the semi-group

$$e^{At}f(x) = e^{(2\lambda_1+\lambda_2)t}\mathbb{E}_{x,+1}\left[f(\widehat{X}_t)\,\middle|\, N_t \text{ even}\right]\frac{1}{2}(1+e^{-2\lambda_1 t})$$
$$- e^{(2\lambda_1+\lambda_2)t}\mathbb{E}_{x,+1}\left[f(\widehat{X}_t)\,\middle|\, N_t \text{ odd}\right]\frac{1}{2}(1-e^{-2\lambda_1 t})$$

Assume $\widehat{X}_t$ has a stationary distribution μ

- $\mathbb{E}_{x,+1}\left[f(\widehat{X}_t)\,\middle|\, N_t \text{ even}\right] = \mu(f) + b_t^e(x)$;
- $\mathbb{E}_{x,+1}\left[f(\widehat{X}_t)\,\middle|\, N_t \text{ odd}\right] = \mu(f) + b_t^o(x)$;
- $b_t^e(x), b_t^o(x) \in O(e^{-\alpha t})$ for some $\alpha \in [0, \lambda_1]$.

The growth rate of the semi-group

$$e^{At}f(x) = e^{(2\lambda_1+\lambda_2)t}\frac{b_t^e(x)-b_t^o(x)}{2} + e^{\lambda_2 t}\left(\mu(f) + \frac{b_t^e(x)+b_t^o(x)}{2}\right).$$

$\Rightarrow \log(e^{At}f(x)) \in O(2\lambda_1+\lambda_2-\alpha)$.

Branching annihilating particles and anti-particles

Setting

- $E^{\uparrow} = \mathbb{N}_0^E \times \mathbb{N}_0^E$;
- $(\eta^+, \eta^-) \in E^{\uparrow}$ configuartion of particles and anti-particles;
- $A = (r(x, y))$ matrix on E satisfying (b) and (c).

Independent movement

- particles and anti-particles jump independently according to rates r_+;
- $x \to y$ at rate $r_+(x, y)$.

Branching and annihilation

- $x \to 2x + (-y)$ at rate $r_-(x, y)$;
- $x + (-x) \to \emptyset$ at rate $\lambda \in [0, \infty]$.

Another stochastic representation

Theorem 3

Assume

- $A = (r(x,y))$ *a generator which may have negative off-diagonal entries;*
- $r(x,x) = -\sum_{y \neq x} r(x,y)$;
- $S_t = e^{tA}$ *semi-group generated by* A.

Then the semi-group has the stochastic representation

$$S_t f(x) = \mathbb{E}_{\delta_x,0} f^{\uparrow}(\eta_t),$$

where

- $\eta_t = (\eta_t^+, \eta_t^-)$ *is the configuration of particles and anti-particles at time* t;
- $f^{\uparrow}((\eta^+, \eta^-)) = \sum_x (\eta^+(x) - \eta^-(x)) f(x)$.

Applied to duality

The signed particle representation

Assume $LH(\cdot,y)(x) = L^{dual}H(x,\cdot)(y)$, but L^{dual} is just an arbitrary matrix. Then, with $(\widehat{X}_t^{dual}, \widehat{U}_t^{dual})$ the signed particle generated by L^{dual},

$$\mathbb{E}_x H(X_t,y) = \mathbb{E}_{y,+1}^{dual}\left(\widehat{U}_t^{dual} H(x,\widehat{X}_t^{dual}) e^{\int_0^t \widehat{V}(\widehat{X}_s^{dual})ds}\right).$$

The branching particle/anti-particle representation

Assume $LH(\cdot,y)(x) = L^{dual}H(x,\cdot)(y)$, but L^{dual} is includes negative rates. Then, with (η_t) the system of particles and anti-particles and $H^{\uparrow}(x,\eta) = \sum_y(\eta^+(y) - \eta^-(y))H(x,y)$,

$$\mathbb{E}_x H^{\uparrow}(X_t,\eta) = \mathbb{E}_\eta^{dual} H^{\uparrow}(x,\eta_t).$$