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Algebraic Duality Methods in Probability

Exact solution of a boundary-driven integrable particle system

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joint work (in progress) with **Rouven Frassek** (ENS Paris)

Introduction

- Symmetric (partial) exclusion process:

$$\mathcal{H} = \sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2S_i^0 S_{i+1}^0 - 2s^2 \right] \quad [s = \frac{1}{2} \text{ integrable}]$$

$\mathfrak{su}(2)$ Lie algebra

$$[S^0, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = 2S^0$$

Introduction

- Symmetric (partial) exclusion process:

$$\mathcal{H} = \sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2S_i^0 S_{i+1}^0 - 2s^2 \right] \quad \left[s = \frac{1}{2} \text{ integrable} \right]$$

$\mathfrak{su}(2)$ Lie algebra

$$[S^0, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = 2S^0$$

- “KMP family” of processes:

$$\mathcal{H} = \sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ - 2S_i^0 S_{i+1}^0 + 2s^2 \right] \quad \left[\text{non-integrable ...} \right]$$

$\mathfrak{su}(1, 1)$ Lie algebra

$$[S^0, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = -2S_0$$

Introduction

► Q1:

Is there a process with **integrable** Hamiltonian and $\mathfrak{su}(1, 1)$ symmetry?

Can we add boundaries?

Answered in [FGK (2020a)] Frassek, G., Kurchan, *Non-compact quantum spin chains as integrable stochastic particle processes*, J. Stat. Phys. 180, 366–397 (2020).

Introduction

► Q1:

Is there a process with **integrable** Hamiltonian and $\mathfrak{su}(1, 1)$ symmetry?
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► Q2:

Can we compute the non-equilibrium steady state?

This is the **subject of this seminar**.

Introduction

- ▶ Emerging properties of systems with **currents** in the stationary state:
 - Long-range correlations
 - Non-Gaussian fluctuations
 - Non-local large deviation functions

- ▶ Q3:
How different are the dynamics of a system evolving in contact with an **equilibrium** bath, from one evolving in contact with a **non-equilibrium** one?

I shall restrict to the **boundary-driven** set-up.

Outline

1. Model

2. Results

- ▶ absorbing dual
- ▶ non-equilibrium steady state

3. Proof

- ▶ algebraic description
- ▶ duality & mapping to equilibrium

4. Perspectives

1. Models

The basic

open symmetric 'harmonic' process

The basic model

Markov process $\{\eta(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{N}_0^N$ with generator

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N$$

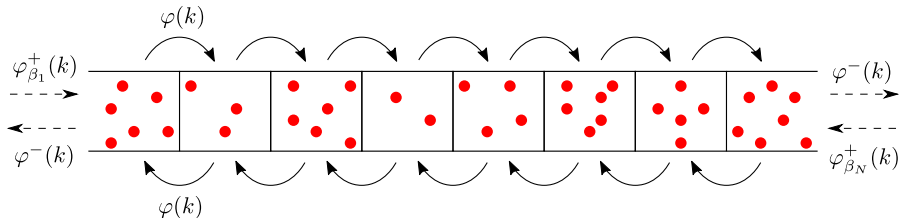
$$\begin{aligned} \mathcal{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{k} \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{1}{k} \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right] \end{aligned}$$

$$\mathcal{L}_1 f(\eta) = \sum_{k=1}^{\eta_1} \frac{1}{k} \left[f(\eta - k\delta_1) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_L^k}{k} \left[f(\eta + k\delta_1) - f(\eta) \right]$$

$$\mathcal{L}_N f(\eta) = \sum_{k=1}^{\eta_N} \frac{1}{k} \left[f(\eta - k\delta_N) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_R^k}{k} \left[f(\eta + k\delta_N) - f(\eta) \right]$$

Remark: $0 < \beta_L, \beta_R < 1$; the holding time in η is an Exp. R.V. with parameter $\sum_{i=1}^N 2h(\eta_i) - \log(1 - \beta_L) - \log(1 - \beta_R)$ $h(n) = \sum_{k=1}^n \frac{1}{k}$ harmonic numbers

The basic model



A chain of length $N = 8$, the i^{th} site corresponds to the i^{th} box.

$$\varphi(k) = \varphi^-(k) = \frac{1}{k} \quad \varphi_{\beta_i}^+(k) = \frac{\beta_i^k}{k}$$

The basic model

- ▶ If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product geometric distribution

$$\mu^{\text{eq}}(\eta) = \prod_{i=1}^N [\beta^{\eta_i} (1 - \beta)] \quad 0 < \beta < 1$$

is reversible, and thus stationary, with density

$$\rho(\beta) = \frac{\beta}{1 - \beta}$$

- ▶ If $\beta_L \neq \beta_R$: *boundary driven non-equilibrium*

$$\mu(\eta) = ?$$

Remark: non-product law (cf. standard zero-range [Levine, Mukamel, Schütz])

The general

open symmetric 'harmonic' process

The general model (spin s)

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N$$

$$\begin{aligned} \mathcal{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi_s(k, \eta_{i+1}) \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right] \end{aligned}$$

$$\mathcal{L}_i f(\eta) = \sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \left[f(\eta - k\delta_i) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} \left[f(\eta + k\delta_i) - f(\eta) \right]$$

$$\varphi_s(k, n) = \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)}$$

$$\psi(z) = \frac{\partial}{\partial z} \log \Gamma(z)$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

$$h_s(n) = \sum_{k=1}^n \varphi_s(k, n) = \psi(n+2s) - \psi(2s) = \sum_{k=1}^n \frac{1}{k+2s-1}$$

The general model (spin s)

- ▶ If $s = \frac{1}{2}$ then we recover the basic model
- ▶ If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product negative-binomial distribution

$$\mu^{\text{eq}}(\eta) = \prod_{i=1}^N \left[\frac{\beta^{\eta_i}}{\eta_i!} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)} (1 - \beta)^{2s} \right] \quad 0 < \beta < 1$$

is reversible.

- ▶ If $\beta_L \neq \beta_R$: *boundary driven* particle system

$$\mu(\eta) = ?$$

Relation to previous models

Relation to previous models

- ▶ The bulk part of the basic model is the $q \rightarrow 1$ limit of the MADM model [Sasamoto-Wadati]

$$\mathcal{L}^{MADM} = \sum_{i \in \mathbb{Z}} \mathcal{L}_{i,i+1}$$

$$\begin{aligned} \mathcal{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{[k]_q} \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{q^k}{[k]_q} \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right] \end{aligned}$$

$$q\text{-number} \quad [k]_q = \frac{1 - q^k}{1 - q} \rightarrow k \quad \text{as } q \rightarrow 1$$

Relation to previous models

- ▶ The bulk part of the spin s model is the $q \rightarrow 1$ limit of the q -Hahn model

[Barraquand-Corwin], [Povolotsky]

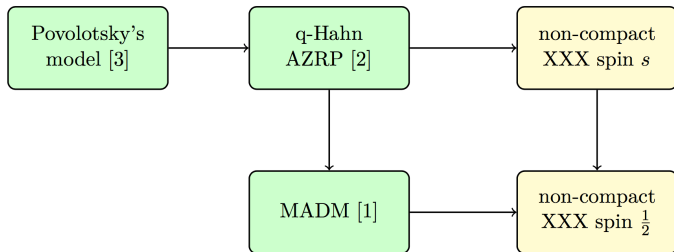
$$\begin{aligned}\mathcal{L}^{q\text{-Hahn}} &= \sum_{i \in \mathbb{Z}} \mathcal{L}_{i,i+1} \\ \mathcal{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi^{r,q,\nu}(k, \eta_i) \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi^{\ell,q,\nu}(k, \eta_{i+1}) \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right]\end{aligned}$$

$$\varphi^{r,q,\nu}(k, n) = \frac{\nu^k (\nu; q)_{n-k} (q; q)_n}{[k]_q (\nu; q)_n (q, q)_{n-k}} \quad \varphi^{\ell,q,\nu}(k, n) = \frac{(\nu; q)_{n-k} (q; q)_n}{[k]_q (\nu; q)_n (q, q)_{n-k}}$$

$$(\nu; q)_n = \prod_{j=0}^{n-1} (1 - \nu q^j) \quad \lim_{q \rightarrow 1} \frac{(q^{2s}; q)_n}{(1-q)^n} = \frac{\Gamma(2s+n)}{\Gamma(2s)}$$

$$\lim_{q \rightarrow 1} \varphi^{r/\ell, q, q^{2s}}(k, n) = \varphi_s(k, n)$$

Relation to previous models



[FGK (2020a)], see also [Frassek '19]

2. Results

Preliminaries:

duality

Duality

Definition [Liggett]

$(\eta_t)_{t \geq 0}$ Markov process on Ω with generator \mathcal{L} ,

$(\xi_t)_{t \geq 0}$ Markov process on Ω_{dual} with generator \mathcal{L}^{dual}

ξ_t is **dual** to η_t with duality function $D : \Omega \times \Omega_{dual} \rightarrow \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is **self-dual** if $\mathcal{L}^{dual} = \mathcal{L}$.

In terms of generators:

$$\mathcal{L}D(\cdot, \xi)(\eta) = \mathcal{L}^{dual}D(\eta, \cdot)(\xi)$$

Duality for Markov chains

Assume state spaces Ω, Ω_{dual} are countable sets,
then the Markov generator L is a matrix $L(\eta, \eta')$ s.t.

$$L(\eta, \eta') \geq 0 \quad \text{if } \eta \neq \eta', \quad \sum_{\eta' \in \Omega} L(\eta, \eta') = 0$$

$$\mathcal{L}D(\cdot, \xi)(\eta) = \mathcal{L}^{dual}D(\eta, \cdot)(\xi)$$

amounts to

$$LD = DL_{dual}^T$$

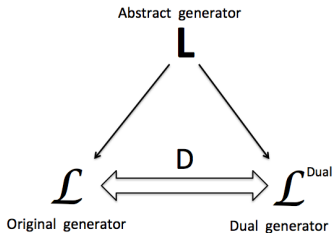
Indeed

$$\sum_{\eta'} L(\eta, \eta')D(\eta', \xi) = \sum_{\xi'} L^{dual}(\xi, \xi')D(\eta, \xi')$$

(Lie) Algebraic approach to duality

Overarching ideas:

- self-duality is derived from symmetries [Schütz]
- duality arises as a change of representation



- The approach is constructive: [Carinci, G., Redig, Sasamoto] quantum groups of rank 1; [Kuan] higher rank quantum groups; [Franceschini, Groenevelt] orthogonal polynomials
- Monograph in preparation with G. Carinci and F. Redig: *'Duality of Markov processes: an algebraic approach'*.

Preliminaries:

factorial moments

Factorial moments

The multivariate factorial moments of order $(\xi_1, \dots, \xi_N) \in \mathbb{N}^N$ of an integer-valued random vector with distribution $\mu(\eta_1, \dots, \eta_N)$ are defined as

$$F(\xi_1, \dots, \xi_N) = \sum_{\eta \in \mathbb{N}^N} \left[\prod_{i=1}^N \eta_i (\eta_i - 1) \cdots (\eta_i - \xi_i + 1) \right] \mu(\eta_1, \dots, \eta_N)$$

Inversion formula

$$\mu(\eta_1, \dots, \eta_N) = \sum_{\xi \in \mathbb{N}^N} F(\xi_1, \dots, \xi_N) \prod_{i=1}^N \binom{\xi_i}{\eta_i} \frac{(-1)^{\xi_i - \eta_i}}{\xi_i!}$$

Factorial moments

For us it will be convenient to consider

$$G(\xi_1, \dots, \xi_N) = \sum_{\eta \in \mathbb{N}^N} \left[\prod_{i=1}^N \frac{\eta_i (\eta_i - 1) \dots (\eta_i - \xi_i + 1)}{2s \ 2s + 1 \ \dots \ (2s + \xi_i - 1)} \right] \mu(\eta_1, \dots, \eta_N)$$

At equilibrium

$$\mu^{\text{eq}}(\eta) = \prod_{i=1}^N \left[\frac{\beta^{\eta_i}}{\eta_i!} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)} (1 - \beta)^{2s} \right] \quad 0 < \beta < 1$$

$$G^{\text{eq}}(\xi_1, \dots, \xi_N) = \prod_{i=1}^N \rho^{\xi_i} \quad \rho := \rho(\beta) = \frac{\beta}{1 - \beta}$$

Results

The dual absorbing process ¹

Markov process $\{\xi(t), t \geq 0\}$ taking values on $\Omega_{N+2} = \mathbb{N}^{N+2}$

Configurations: $\xi = (\xi_0, \xi_1, \dots, \xi_N, \xi_{N+1}) \in \Omega_{N+2}$

Generator: $\mathcal{L}^{dual} = \mathcal{L}_{0,1}^{\leftarrow} + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_{N,N+1}^{\rightarrow}$

$$\mathcal{L}_{0,1}^{\leftarrow} f(\xi) = \sum_{k=1}^{\xi_1} \varphi(k, \xi_1) \left[f(\xi + k\delta_0 - k\delta_1) - f(\xi) \right]$$

$$\mathcal{L}_{N,N+1}^{\rightarrow} f(\xi) = \sum_{k=1}^{\xi_N} \varphi(k, \xi_N) \left[f(\xi - k\delta_N + k\delta_{N+1}) - f(\xi) \right]$$

¹Spohn, Presutti, De Masi ; for a recent review see [Carinci, G., Giberti, Redig]

Duality of the open symmetric harmonic process

Theorem [FGK (2020a), FG (2021+)]

1. The open symmetric harmonic process $\{\eta(t), t \geq 0\}$ with generator \mathcal{L} and the absorbing process $\{\xi(t), t \geq 0\}$ with generator \mathcal{L}^{dual} are dual

$$D(\eta, \xi) = \rho_L^{\xi_0} \left[\prod_{i=1}^N \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)} \right] \rho_R^{\xi_{N+1}}$$
$$\rho_L = \frac{\beta_L}{1 - \beta_L} \quad \rho_R = \frac{\beta_R}{1 - \beta_R}$$

Namely

$$\mathbb{E}_\eta [D(\eta(t), \xi)] = \mathbb{E}_\xi [D(\eta, \xi(t))]$$

Factorial moments and absorption probabilities

Theorem

2. Let $G(\xi_1, \dots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ_N .

Consider dual configuration $\check{\xi} = (0, \xi_1, \dots, \xi_N, 0)$ with $|\xi| = \sum_{i=1}^N \xi_i$ particles.

Then

$$G(\xi_1, \dots, \xi_N) = \sum_{k=0}^{|\xi|} \rho_L^k \rho_R^{|\xi|-k} p_{\check{\xi}}(k)$$

where

$$p_{\check{\xi}}(k) = \mathbb{P} \left[\xi(\infty) = k\delta_0 + (|\xi| - k)\delta_{N+1} \mid \xi(0) = \check{\xi} \right]$$

absorption probabilities

Factorial moments: explicit expression I

Theorem [FG (2021+)]

3. Let $G(\xi_1, \dots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ_N .
Then [assuming $2s \in \mathbb{N}$]

$$G(\xi_1, \dots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_\xi(n)$$

with

$$g_\xi(n) = \sum_{\substack{n_1, \dots, n_N \\ \sum_j n_j = n}} \prod_{i=1}^N \binom{\xi_i}{n_i} \prod_{j=1}^{2s} \frac{2s(N+2-i) - j}{2s(N+2-i) - j + \sum_{k=j}^N n_k}$$

In other words

$$p_\xi(k) = \sum_{n=k}^{|\xi|} (-1)^{n-k} \binom{n}{k} g_\xi(n)$$

Factorial moments: explicit expression II

Theorem [FG (2021+)]

4. If we identify $\xi = (\xi_1, \dots, \xi_N)$ with the ordered set $x = (x_1, x_2, \dots, x_{|\xi|})$ with $1 \leq x_1 \leq x_2 \leq \dots \leq x_{|\xi|} \leq N$, then

$$G(\xi_1, \dots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_x(n)$$

with

$$g_x(n) = \sum_{1 \leq i_1 < \dots < i_n \leq |\xi|} \prod_{\alpha=1}^n \frac{n - \alpha + 2s(N + 1 - x_{i_\alpha})}{n - \alpha + 2s(N + 1)}$$

Mapping to equilibrium

Theorem [FGK(2020b); FG (2021+)]

5. As a consequence of duality, there exists a matrix P such that

$$H^{eq} = P^{-1}HP$$

where

$$H^{eq} = (L^{eq})^T \quad H = L^T$$

- The mapping P was observed macroscopically by [Tailleur, Kurchan, Lecomte] in the context of the MFT [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim]

MACRO:

Local Equilibrium & Fick's law

Local equilibrium & Fick's law

Let μ_N be the invariant measure.

(i) For all f bounded cylindrical function on $\mathbb{N}_0^{\mathbb{N}}$

$$\lim_{N \rightarrow \infty} \mu_N(\tau_{[uN]} f) = \nu_{\rho(u)}(f) \quad u \in (0, 1)$$

where

- τ_i translation by i
- ν_{ρ} is the product measure on $\mathbb{N}_0^{\mathbb{N}}$ with marginals NegBin $(2s, \rho)$
- $\rho(u) = \rho_L + (\rho_R - \rho_L)u$

(ii) Fick's law

$$J = -K_s \frac{d\rho(u)}{du} \quad u \in (0, 1)$$

where

- $K_s = 2s$
- $J = \lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} \left[\sum_{\eta \in \mathbb{N}_0^{\mathbb{N}}} \mu_N(\eta) [\eta_i - \eta_{i+1}] \right]$

Proof of local equilibrium

Given $(\xi_1, \dots, \xi_N) \in \mathbb{N}^N$ made of $|\xi|$ dual particles located at $1 \leq x_1 \leq \dots \leq x_{|\xi|} \leq N$

define $(\xi_1^u, \dots, \xi_N^u) \in \mathbb{N}^N$ with dual particles located at $x_1 + [uN] \leq \dots \leq x_{|\xi|} + [uN]$

Then we need to prove that

$$\lim_{N \rightarrow \infty} G(\xi_1^u, \dots, \xi_N^u) = [\rho(u)]^{|\xi|}$$

The explicit formula gives

$$\lim_{N \rightarrow \infty} g_{x^u}(n) = \binom{|\xi|}{n} (1-u)^n$$

and therefore

$$\lim_{N \rightarrow \infty} G(\xi_1^u, \dots, \xi_N^u) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n \binom{|\xi|}{n} (1-u)^n = [\rho(u)]^{|\xi|}$$

MICRO:

Correlation functions

1 dual particle : $\xi = \delta_x$

$$D(\eta, \delta_x) = \eta_x$$

$$\mathbb{E}[\eta_x] = \rho_L p_x(1) + \rho_R p_x(0)$$

$$p_x(1) = 1 - \frac{i}{N+1} \qquad p_x(0) = \frac{i}{N+1}$$

$$\rho_x := \langle \eta_x \rangle = \rho_L + \frac{\rho_R - \rho_L}{N+1} x \qquad \text{Linear profile}$$

$$J := -\langle \eta_{x+1} - \eta_x \rangle = -\frac{\rho_R - \rho_L}{N+1} \qquad \text{Fick's law}$$

2 dual particles : $\xi = \delta_{x_1} + \delta_{x_2}$

$$\text{Cov}(\eta_{x_1}, \eta_{x_2}) = \mathbb{E}[\eta_{x_1} \eta_{x_2}] - \mathbb{E}[\eta_{x_1}] \mathbb{E}[\eta_{x_2}]$$

$$G_{x_1, x_2} = \mathbb{E}[D(\eta, \delta_{x_1} + \delta_{x_2})] \quad G_{x_1} = \mathbb{E}[D(\eta, \delta_{x_1})]$$

$$\text{Cov}(\eta_{x_1}, \eta_{x_2}) = \begin{cases} G_{x_1, x_2} - G_{x_1} G_{x_2} & \text{if } x_1 \neq x_2 \\ 2G_{x_1, x_1} + G_{x_1} [1 - G_{x_1}] & \text{if } x_1 = x_2 \end{cases}$$
$$= \frac{x_1(N+1-x_2)}{(N+1)^2(N+2)} (\rho_L - \rho_R)^2 \quad \text{if } x_1 \neq x_2$$

Remark: Long range correlations

$$\lim_{N \rightarrow \infty} N \text{Cov}(\eta_{y_1 N}, \eta_{y_2 N}) = y_1(1-y_2)(\rho_L - \rho_R)^2 \quad 0 \leq y_1 < y_2 \leq 1$$

3 dual particles : $\xi = \delta_{x_1} + \delta_{x_2} + \delta_{x_3}$

$$\begin{aligned}\kappa_3(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) &= \mathbb{E}[\eta_{x_1} \eta_{x_2} \eta_{x_3}] + \mathbb{E}[\eta_{x_1}] \mathbb{E}[\eta_{x_2}] \mathbb{E}[\eta_{x_3}] \\ &\quad - \mathbb{E}[\eta_{x_1} \eta_{x_2}] \mathbb{E}[\eta_{x_3}] - \mathbb{E}[\eta_{x_1} \eta_{x_3}] \mathbb{E}[\eta_{x_2}] - \mathbb{E}[\eta_{x_2} \eta_{x_3}] \mathbb{E}[\eta_{x_1}]\end{aligned}$$

Using three dual particles started at $1 \leq x_1 < x_2 < x_3 \leq N$

$$\kappa_3(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) = \frac{2x_1(N+1-2x_2)(N+1-x_3)}{(N+1)^3(N+2)(N+3)} (\rho_R - \rho_L)^3$$

Remark: for $0 \leq y_1 < y_2 < y_3 \leq 1$

$$\lim_{N \rightarrow \infty} N^2 \kappa_3(\eta_{y_1 N}, \eta_{y_2 N}, \eta_{y_3 N}) = 2y_1(1-2y_2)(1-y_3)(\rho_R - \rho_L)^3$$

$$N^{n-1} \kappa_n \sim f_n(y_1, \dots, y_n) (\rho_R - \rho_L)^n$$

proved by orthogonal duality in [Floreani, Redig, Sau '20]

Comparison to inhomogeneous product measure

Fluctuation of total mass $|\eta| = \sum_{x=1}^N \eta_x$

$\mathbb{E}[\cdot]$ non-equilibrium steady state

$\mathbb{E}_{loc}[\cdot]$ inhomog. product state

$$\otimes_{x=1}^N \text{NegBin}(2s, \rho_x)$$

► average

$$\mathbb{E}[|\eta|] = \mathbb{E}_{loc}[|\eta|] = \sum_{x=1}^N \rho_x = \sum_{x=1}^N 2s \left(\rho_L + \frac{\rho_R - \rho_L}{N+1} x \right) = 2sN \left(\frac{\rho_L + \rho_R}{2} \right)$$

► fluctuations

$$\text{Var}[|\eta|] = \sum_{x_1, x_2=1}^N \text{Cov}[\eta_{x_1}, \eta_{x_2}]$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Var}[|\eta|] = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Var}_{loc}[|\eta|] + \frac{s}{6} (\rho_L - \rho_R)^2$$

For $s = -\frac{1}{2}$ one recovers open SEP [Derrida, Lebowitz, Speer, PRL01]

3. Proof (Ideas)

Algebraic description

$\mathfrak{su}(1, 1)$ Lie algebra

Non-compact Lie algebra with generators satisfying

$$[S^0, S^\pm] = \pm S^\pm \quad [S^+, S^-] = -2S^0$$

Representation with ∞ -dimensional matrices (spin $s > 0$)

$$S^+|n\rangle = (2s + n)|n+1\rangle$$

$$S^-|n\rangle = n|n-1\rangle$$

$$S^0|n\rangle = (n + s)|n\rangle$$

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & & & & & \\ & 2s & & & & \\ & & \ddots & & & \\ & & & 2s+1 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 1 & & & & \\ & & \ddots & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix} \quad S^0 = \begin{pmatrix} s & 0 & & & & \\ & s+1 & & & & \\ 0 & & \ddots & & & \\ & & & s+2 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}$$

Integrable Hamiltonian: open XXX chain with $\mathfrak{su}(1, 1)$ spins

$$H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

- ▶ Bulk: [Beisert, Faddeev, Korchemsky, Derkachov ...]

$$H_{i,i+1} = 2 \left(\psi(S_{i,i+1}) - \psi(2s) \right)$$

$$S_{i,i+1}(S_{i,i+1} - 1) = S_i^0 S_{i+1}^0 - \frac{1}{2} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

- ▶ Boundary [FGK(2020a)]

$$H_1 = e^{-S_1^-} e^{\rho_L S_1^+} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{-\rho_L S_1^+} e^{S_1^-}$$

$$H_N = e^{-S_N^-} e^{\rho_R S_N^+} \left(\psi(S_N^0 + s) - \psi(2s) \right) e^{-\rho_R S_N^+} e^{S_N^-}$$

In the discrete representation

$$L = H^T$$

Integrable Hamiltonian: alternative form of the bulk

$$H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

► Bulk: [FG (2021+)]

$$H_{i,i+1} = H_{i,i+1}^{\rightarrow} + H_{i,i+1}^{\leftarrow}$$

$$H_{i,j}^{\rightarrow} = e^{-S_j^+(S_j^0+s)^{-1}S_i^-} \left(\psi(S_i^0 + s) - \psi(2s) \right) e^{S_j^+(S_j^0+s)^{-1}S_i^-}$$

$$H_{i,j}^{\leftarrow} = e^{-S_i^+(S_i^0+s)^{-1}S_j^-} \left(\psi(S_j^0 + s) - \psi(2s) \right) e^{S_i^+(S_i^0+s)^{-1}S_j^-}$$

Duality explained

$$LD = DL_{dual}^T$$



$$H^T D = DH_{dual}$$



$$(H_1 + \mathbf{H} + H_N)^T D = D (H_{0,1}^{\leftarrow} + \mathbf{H} + H_{N,N+1}^{\rightarrow})$$

Bulk (self-)duality: $\mathbf{H}^T D = D \mathbf{H}$

► Bulk Hamiltonian $\mathfrak{su}(1, 1)$ symmetry:

$$\mathbf{H} := \sum_{i=1}^{N-1} H_{i,i+1} \quad \mathbf{S}^+ := \sum_{i=1}^N S_i^+$$

$$[\mathbf{H}, \mathbf{S}^0] = [\mathbf{H}, \mathbf{S}^+] = [\mathbf{H}, \mathbf{S}^-] = 0$$

► Reversibility

$$\mathbf{H}^T \mathbf{d} = \mathbf{d} \mathbf{H}$$

$$\mathbf{d}(\eta, \xi) = \prod_{i=1}^N \eta_i! \frac{\Gamma(2s)}{\Gamma(\eta_i + 2s)} \delta_{\eta, \xi}$$

► Bulk (self-)duality

$$\mathbf{H}^T \mathbf{d} \mathbf{e}^{\mathbf{S}^+} = \mathbf{d} \mathbf{H} \mathbf{e}^{\mathbf{S}^+} = \mathbf{d} \mathbf{e}^{\mathbf{S}^+} \mathbf{H}$$

$$\mathbf{d} \mathbf{e}^{\mathbf{S}^+}(\eta, \xi) = \prod_{i=1}^N \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)}$$

Boundary duality: $H_1^T D = D H_{0,1}^{\leftarrow}$

$$H_1 = e^{-S_1^-} e^{\rho_L S_1^+} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{-\rho_L S_1^+} e^{S_1^-}$$

$$H_1^T = d_1 e^{S_1^+} e^{-\rho_L S_1^-} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{\rho_L S_1^-} (d_1 e^{S_1^+})^{-1}$$

Boundary duality: $H_1^T D = D H_{0,1}^{\leftarrow}$

$$H_1 = e^{-S_1^-} e^{\rho_L S_1^+} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{-\rho_L S_1^+} e^{S_1^-}$$

$$H_1^T = d_1 e^{S_1^+} e^{-\rho_L S_1^-} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{\rho_L S_1^-} (d_1 e^{S_1^+})^{-1}$$

Two representations of $\mathfrak{su}(1, 1)$ with intertwiner $l_0(\rho_L, \xi_0) = \rho_L^{\xi_0}$

$$\left\{ \begin{array}{l} S_0^+ = \rho_L \left(\rho_L \frac{\partial}{\partial \rho_L} + 2s \right) \\ S_0^- = \frac{\partial}{\partial \rho_L} \\ S_0^0 = \left(\rho_L \frac{\partial}{\partial \rho_L} + s \right) \end{array} \right. \quad \left\{ \begin{array}{l} S_0^+ |\xi_0\rangle = (\xi_0 + 2s) |\xi_0 + 1\rangle \\ S_0^- |\xi_0\rangle = \xi_0 |\xi_0 - 1\rangle \\ S_0^0 |\xi_0\rangle = (\xi_0 + s) |\xi_0\rangle \end{array} \right.$$

$$B_{0,1} = e^{-s_0^+(s_0^0+s)^{-1}s_1^-} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{s_0^+(s_0^0+s)^{-1}s_1^-}$$

$$S_0^+(S_0^0 + s)^{-1} = \rho_L$$

$$S_0^+(S_0^0 + s)^{-1} |m_0\rangle = |m_0 + 1\rangle$$

$$B_{0,1} = (d_1 e^{S_1^+})^{-1} H_1^T (d_1 e^{S_1^+})$$

$$B_{0,1} = H_{0,1}^{\leftarrow}$$

$$\mathcal{B} l_0 = l_0 \mathcal{B} \quad \implies \quad H_1^T (l_0 d_1 e^{S_1^+}) = (l_0 d_1 e^{S_1^+}) H_{0,1}^{\leftarrow}$$

Isospectrality

▶ $H = H_1 + \mathbf{H} + H_N$

$$H_i = e^{-S_i^-} e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s) \right) e^{-\rho_i S_i^+} e^{S_i^-}$$

▶ $H' = e^{S^-} H e^{-S^-} = H'_1 + \mathbf{H} + H'_N$

$$H'_i = e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s) \right) e^{-\rho_i S_i^+} \quad \text{upper triangular}$$

▶ $H^\circ = H_1^\circ + \mathbf{H} + H_N^\circ$

$$H_i^\circ = \psi(S_i^0 + s) - \psi(2s) \quad \text{block diagonal}$$

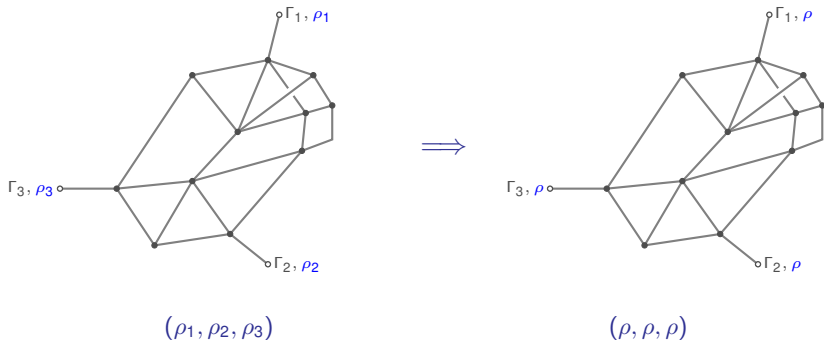
They all have the same spectrum (which is independent of ρ_L, ρ_R)!

As a consequence there exists P such that

$$H^{eq} = P^{-1} H P$$

Mapping to equilibrium

The mapping holds on any graph, with multiple reservoirs



$$H^{eq} = P^{-1} H P$$

Sequence of similarity transformations

► $H' = e^{S^-} H e^{-S^-}$

$$H'_1 = e^{\rho_L S_1^+} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{\rho_L S_1^+} \quad H'_N = e^{\rho_R S_N^+} \left(\psi(S_N^0 + s) - \psi(2s) \right) e^{\rho_R S_N^+}$$

► $H'' = e^{-\rho_R S^+} H' e^{\rho_R S^+}$

$$H''_1 = e^{(\rho_L - \rho_R) S_1^+} \left(\psi(S_1^0 + s) - \psi(2s) \right) e^{(\rho_L - \rho_R) S_1^+} \quad H''_N = \psi(S_N^0 + s) - \psi(2s)$$

► $H^\circ = W^{-1} H'' W$

$$H^\circ_1 = \psi(S_1^0 + s) - \psi(2s)$$

$$H^\circ_N = \psi(S_N^0 + s) - \psi(2s)$$

Non-trivial symmetries (QIS method)

- ▶ First non-local charge:

$$[H'', Q''] = 0$$

$$Q'' = Q^\circ - (\rho_L - \rho_R)Q^+$$

$$Q^\circ = \mathbf{S}^0 (\mathbf{S}^0 + 2s - 1)$$

$$Q^+ = s\mathbf{S}^+ + \sum_{i=1}^N S_i^+ \left(S_i^0 + 2 \sum_{j=i+1}^N S_j^0 \right)$$

- ▶ To find W we rather solve

$$Q^\circ = W^{-1}Q''W$$

- ▶ Ansatz:

$$W = 1 + \sum_{k=1}^{\infty} (\rho_L - \rho_R)^k w_k \quad \Longrightarrow \quad [Q^\circ, w_k] = Q^+ w_{k-1}$$

- ▶ This difference equation can be solved

$$W = \sum_{k=0}^{\infty} (\rho_L - \rho_R)^k \frac{(Q^+)^k}{k!} \frac{\Gamma(2(\mathbf{S}^0 + s))}{\Gamma(k + 2(\mathbf{S}^0 + s))}$$

non-local!

Back tracking

- ▶ $H^0|\Omega\rangle = 0$ with $|\Omega\rangle = |0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle$
- ▶ $H''|g''\rangle = 0$ with $|g''\rangle = W|\Omega\rangle$
- ▶ $H'|g'\rangle = 0$ with $|g'\rangle = e^{\rho R S^+}|g''\rangle$
- ▶ $H|g\rangle = 0$ with $|g\rangle = e^{-S^-}|g'\rangle$

- ▶ Mapping to equilibrium

$$P = e^{-S^-} e^{\rho R S^+} W e^{-\rho S^+} e^{S^-}$$

4. Perspectives

The basic model in integral form ($s=1/2$)

Lévy process $\{z(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{R}_+^N$

$z_i(t) \equiv$ energy at site $i \in \{1, 2, \dots, N\}$ at time $t \geq 0$

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N$$

$$\begin{aligned} \mathcal{L}_{i,i+1} f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \left[f(z - \alpha\delta_i + \alpha\delta_{i+1}) - f(z) \right] \\ &+ \int_0^{z_{i+1}} \frac{d\alpha}{\alpha} \left[f(z + \alpha\delta_i - \alpha\delta_{i+1}) - f(z) \right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}_i f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \left[f(z - \alpha\delta_i) - f(z) \right] \\ &+ \int_0^\infty d\alpha \frac{e^{-\lambda_i \alpha}}{\alpha} \left[f(z + \alpha\delta_i) - f(z) \right] \end{aligned}$$

Conclusions

- ▶ Reasoning with Lie algebras is useful for duality and its consequences.
- ▶ Integrability (on top of duality) opens up the possibility of explicit formulae for boundary-driven models, connecting non-equilibrium to equilibrium.
- ▶ Open problems:
 - fluctuations
 - matrix product ansatz
 - open chain with $q \neq 1$
 - higher rank non-compact algebras
- ▶ The algebraic approach to duality can be extended to quantum systems (e.g. FGK(2021), arXiv:2008.03476, quantum symmetric exclusion process).

Thank you for your attention