1. The terms of a series are defined recursively by the equations

\[ a_1 = 2, \quad a_{n+1} = \frac{5n + 1}{4n + 3} a_n. \]

Determine whether the series \( \sum a_n \) converges or diverges.

**Method 1:**

Note that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{5n + 1}{4n + 3} a_n}{a_n} \right| = \lim_{n \to \infty} \left| \frac{5n + 1}{4n + 3} \right| = \frac{5}{4} > 1.
\]

Therefore, by the Ratio Test, the series diverges.

**Method 2:**

Note that if \( n \geq 2 \), then \( 5n - 4n \geq 3 - 1 \implies 5n + 1 \geq 4n + 3 \). Therefore, if \( n \geq 2 \),

\[ \frac{5n + 1}{4n + 3} \geq 1 \implies \frac{5n + 1}{4n + 3} a_n \geq a_n \implies a_{n+1} \geq a_n. \]

Note also that

\[ a_2 = \frac{6}{7} a_1 = \frac{12}{7}. \]

Therefore, every term of the sequence \( (a_n) \) is greater than or equal to \( 2 \). Therefore, either

\[ \lim_{n \to \infty} a_n = \infty \text{ or } \lim_{n \to \infty} a_n \geq \frac{12}{7}. \]

Therefore,

\[ \lim_{n \to \infty} a_n \neq 0. \]

Thus, by the Test for Divergence, \( \sum a_n \) diverges.
2. Determine whether the following series diverges or converges.

\[ \sum_{n=1}^{\infty} \frac{(n + 1) 5^n}{n3^{2n}} \]

**Method 1:**

Note that

\[ \lim_{n \to \infty} \left| \frac{(n + 2) 5^{n+1}}{(n + 1) 3^{2n+2}} \cdot \frac{n3^{2n}}{(n + 1) 5^n} \right| = \frac{5}{9} < 1. \]

Therefore, by the Ratio Test, the series converges.

**Method 2:**

Note that

\[ \lim_{n \to \infty} \sqrt[n]{\frac{(n + 1) 5^n}{n3^{2n}}} = \lim_{n \to \infty} \sqrt[n]{\frac{n + 1}{n}} \cdot \sqrt[n]{\left(\frac{5}{9}\right)^n} = \frac{5}{9} < 1. \]

Therefore, by the Root Test, the series converges.

**Method 3:**

Note that

\[ \sum_{n=1}^{\infty} \frac{(n + 1) 5^n}{n3^{2n}} = \sum_{n=1}^{\infty} \frac{(n + 1)}{n} \cdot \left(\frac{5}{9}\right)^n. \]

Also,

\[ \lim_{n \to \infty} \left[ \frac{(n+1)}{n} \cdot \left(\frac{5}{9}\right)^n \right] = 1. \]

Recall that since \( \frac{5}{9} < 1 \), that \( \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^n \) is a convergent geometric series. Therefore by the Limit Comparison Test, the original series converges.
3. Find the radius of convergence and interval of convergence of the following series.

\[ \sum_{n=1}^{\infty} \frac{(x - 4)^n}{n5^n} \]

First, note that the center of the interval of convergence is 4. We now use the ratio test to determine the radius of convergence of the series.

\[
\lim_{n \to \infty} \left| \frac{(x - 4)^{n+1}}{(n + 1)5^{n+1}} \cdot \frac{n5^n}{(x - 4)^n} \right| = \lim_{n \to \infty} \left| \frac{n}{n + 1} \cdot \frac{1}{5} |x - 4| \right| = \frac{1}{5} |x - 4|.
\]

Since the ratio test applies only if the limit above is less than 1, and

\[ \frac{1}{5} |x - 4| < 1 \implies |x - 4| < 5, \]

the radius of convergence of the series is 5.

Hence, if \( x \in (4 - 5, 4 + 5) = (-1, 9) \), then the series converges. However, we have not found the interval of convergence yet, because we have to determine the behavior of the series at the endpoints of the interval.

First we check what happens if \( x = -1 \):

\[ \sum_{n=1}^{\infty} \frac{(x - 4)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n} \left( -\frac{5}{5} \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}. \]

This is the alternating harmonic series, which converges by the alternating series test. Therefore, the series converges when \( x = -1 \). Now we check what happens if \( x = 9 \):

\[ \sum_{n=1}^{\infty} \frac{(x - 4)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{5}{5} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n}. \]

This is the harmonic series, which diverges by the \( p \)-test. Therefore, the series diverges when \( x = 9 \). This gives us the interval of convergence \((-1, 9)\).