TENSORS WITH MAXIMAL SYMMETRIES

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Abstract. We classify tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with maximal and next to maximal dimensional symmetry groups under a natural genericity assumption (1-genericity), for $m \geq 14$. In other words, we classify minimal dimensional $GL_m^{\times 3}$-orbits in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ assuming 1-genericity. Our study uncovers new tensors with striking geometry. This paper was motivated by Strassen's laser method for bounding the exponent of matrix multiplication. The best known tensor for the laser method is the large Coppersmith-Winograd tensor, and our study began with the observation that it has a large symmetry group, of dimension $\binom{m+1}{2}$. We show that in odd dimensions, this is the largest possible for a 1-generic tensor, but in even dimensions we exhibit a tensor with a larger dimensional symmetry group. In the course of the proof, we classify nondegenerate bilinear forms with large dimensional stabilizers, which may be of interest in its own right.

1. Introduction

This article studies tensors $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with large symmetry groups, i.e., small $GL_m^{\times 3}$-orbits in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$. The smallest orbits in such a tensor space under this action are classically known. Our study is primarily motivated by the complexity of matrix multiplication, and in this context one imposes a natural genericity condition on the tensors of interest. This brings into play new small orbits with unexpectedly rich geometric structure.

Besides their relevance for computer science, our results are connected to a classical question in algebraic geometry and representation theory: given a representation V of a group G, what are the vectors $v \in V$ whose orbit closures are of small dimension, i.e., with large stabilizers? Our main result (Theorem A) fits into a long tradition of studying small orbits; see for instance [18, 13, 19, 29, 17].

The *exponent of matrix multiplication* ω is a fundamental constant that controls the complexity of basic operations in linear algebra. It is generally conjectured that $\omega = 2$, which would imply that one could multiply $\mathbf{n} \times \mathbf{n}$ matrices using $O(\mathbf{n}^{2+\epsilon})$ arithmetic operations for any $\epsilon > 0$. The current state of knowledge is $2 \leq \omega \leq 2.3728639$ [22] but it has been known since 1989 that $\omega \leq 2.3755$ [15].

One motivation for this paper is the *Ambainis-Filmus-Le Gall challenge*: find new tensors that give good upper bounds on ω via Strassen's laser method [27]. (See [15, 8, 3] for expositions of the method.) This challenge is motivated by the results of [3], where the authors showed

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that the main tool used so far to obtain upper bounds, Strassen's laser method applied to the Coppersmith-Winograd tensor using coordinate restrictions, can never prove $\omega < 2.3$. (Also see [1, 2, 11] for further limitations.) Tensors with continuous symmetry are central to the implementation of the laser method. Advancing ideas in [21], we isolate geometric features of the Coppersmith-Winograd tensors and find other tensors with similar features, in the hope they will be useful for the laser method. The point of departure of this paper was the observation that Coppersmith-Winograd tensors have very large symmetry groups. This led us to the classification problem. Our main theorem, while uncovering new geometry, fails to produce new tensors good for the laser method, as none of the tensors in Theorem A is better than the big Coppersmith-Winograd tensor for the laser method. However, in [14], guided by the results in this paper, we introduce a skew cousin of the little Coppersmith-Winograd tensor $T_{cw,q}$, analyze its utility for the laser method, and show it is potentially better for the laser method than existing tensors. In particular, $T_{skewcw,2}$, like its cousin $T_{cw,2}$, potentially could be used to prove $\omega = 2$.

The largest possible symmetry group for any tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m =: A \otimes B \otimes C$ is realized by a rank one tensor, i.e., an element of the form $a \otimes b \otimes c$.

A tensor $T \in A \otimes B \otimes C$ is *concise* if the induced maps $T_A : A^* \to B \otimes C$, $T_B : B^* \to A \otimes C$, $T_C : C^* \to A \otimes B$ are injective. In our main theorem, we will require additional natural genericity conditions that date back to [28] and have been recently studied in [6, 21, 12]. A tensor is 1_A-*generic* if the subspace $T_A(A^*) \subset B \otimes C$ contains an element of maximal rank; 1_A, 1_B or 1_C-generic tensors are essentially those for which Strassen's equations [26] are non-trivial. A tensor is *binding* if it is 1_A and 1_B-generic. As observed in [6], binding tensors are exactly the structure tensors of unital (not necessarily associative) algebras. Binding tensors are automatically concise. A tensor is 1-*generic* if it is $1_A, 1_B$ and 1_C generic. (1-genericity is called *bequem* in [28] and *comfortable* in [12].) Propositions 3.1 and 3.2 respectively determine the maximum possible dimension of the symmetry group of a 1_A-generic tensor and a binding tensor and show in each case that there is a unique such tensor with maximal dimensional symmetry.

Our main result (Theorem A) classifies 1-generic tensors with symmetry group of maximal and next to maximal dimension. In particular, when m is even, there is a striking gap in that the second largest symmetry group has dimension $m-2$ less than the largest. Of independent interest is Lemma 5.1, which determines which non-degenerate bilinear forms on $\mathbb{C}^k$ have stabilizers of dimension at least $\frac{k^2}{2} - \frac{3k}{2}$.

Notations and conventions. Let $a_1, \ldots, a_m$ be a basis of the vector space A, and $\alpha^1, \ldots, \alpha^m$ its dual basis in A^*. Similarly $b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ are bases of B and C respectively, with corresponding dual bases $\beta^1, \ldots, \beta^m$ and $\gamma^1, \ldots, \gamma^m$. Informally, the symmetry group of a tensor $T \in A \otimes B \otimes C$ is its stabilizer under the natural action of $GL(A) \times GL(B) \times GL(C)$. For a tensor $T \in A \otimes B \otimes C$, let G_T denote its symmetry group. We say T' is *isomorphic* to T if they are in the same $GL(A) \times GL(B) \times GL(C)$-orbit, and generally identify isomorphic tensors. Since the action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$ is not faithful, we work modulo the kernel of its inclusion into $GL(A \otimes B \otimes C)$, which is a 2-dimensional abelian subgroup. (See §2 for details.) The transpose of a matrix M is denoted $M^{\mathbf{t}}$. For a set X, $\overline{X}$ denotes its Zariski closure. For a subset $Y \subset \mathbb{C}^N$, we let $\langle Y \rangle \subset \mathbb{C}^N$ denote its linear span. Throughout we use the summation convention: *indices appearing up and down are to be summed over their range.* Index ranges employed throughout the article are:

$$1 \leq i, i', j, j', k, k', l \leq m,$$
$$2 \leq \rho, \rho', \sigma, \sigma', \tau, \tau' \leq m,$$
$$2 \leq s, t, u \leq m-1.$$

Theorem A. *Let $m \geq 14$ and let $\dim A = \dim B = \dim C = m$.*

Let m be even:

(1) *There is a unique up to isomorphism 1-generic tensor $T \in A \otimes B \otimes C$ such that $\dim G_T = \frac{m^2}{2} + \frac{3m}{2} - 2$, namely*

$$T_{max,even,m} = T_{skewCW,m-2} = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho + \left[\sum_{\xi=2}^{\frac{m}{2}} (a_\xi \otimes b_{\xi+\frac{m}{2}-1} - a_{\xi+\frac{m}{2}-1} \otimes b_\xi)\right] \otimes c_m.$$

(2) *There is a unique up to isomorphism 1-generic tensor $T \in A \otimes B \otimes C$ such that $\dim G_T = \frac{m^2}{2} + \frac{m}{2}$, namely the big Coppersmith–Winograd tensor*

$$T_{max-(m-2),even,m} = T_{CW,m-2} = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho + \left[\sum_{s=2}^{m-1} a_s \otimes b_s\right] \otimes c_m.$$

(3) *All other 1-generic tensors $T \in A \otimes B \otimes C$ satisfy $\dim G_T < \frac{m^2}{2} + \frac{m}{2}$.*

Let m be odd:

(1) *There are exactly two 1-generic tensors $T \in A \otimes B \otimes C$ up to isomorphism such that $\dim G_T = \frac{m^2}{2} + \frac{m}{2}$, namely*

$$T_{max,odd,skew,m} = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho + \left[a_2 \otimes b_2 + \sum_{\xi=3}^{\frac{m-3}{2}} \left(a_\xi \otimes b_{\xi+p-1} - a_{\xi+p-1} \otimes b_\eta\right)\right] \otimes c_m,$$

and

$$T_{max,odd,sym,m} = T_{CW,m-2} = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho + \left[\sum_{s=2}^{m-1} a_s \otimes b_s\right] \otimes c_m.$$

(2) *There is a unique up to isomorphism 1-generic tensor $T \in A \otimes B \otimes C$ such that $\dim G_T = \frac{m^2}{2} + \frac{m}{2} - 1$ and it is*

$$T_{max-1,odd,m} = a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho + \left[\sum_{\xi=2}^{p+1} (a_\xi \otimes b_{\xi+p} - a_{\xi+p} \otimes b_\xi)\right] \otimes c_1.$$

(3) *All other 1-generic tensors $T \in A \otimes B \otimes C$ satisfy $\dim G_T < \frac{m^2}{2} + \frac{m}{2} - 1$.*

The tensors (1) in both cases and (2) in the even case may all be written naturally as elements of $A \otimes A \otimes A^*$ as follows. Let σ_{12} be the permutation switching the first two factors, write

$A = L_1 \oplus N \oplus L_m$, where $L_1 = \langle a_1 \rangle$, $N = \langle a_2, ..., a_{m-1} \rangle$, and $L_m = \langle a_m \rangle$, and let $\mathcal{B} \in N \otimes N$ be a non-degenerate bilinear form. Then

$$T = a_1 \otimes a_1 \otimes \alpha^1 + a_1 \otimes \mathrm{Id}_N + \sigma_{12}(a_1 \otimes \mathrm{Id}_N) + \mathcal{B} \otimes \alpha^m.$$

The symmetry group of $\mathcal{B}$, which we denote $H_{\mathcal{B}} \subset GL(N)$, is naturally contained in G_T. When T is the Coppersmith-Winograd tensor, $\mathcal{B}$ is symmetric and so $H_{\mathcal{B}}$ is an orthogonal group. For the tensor $T_{max,even,m}$, $\mathcal{B}$ is skew-symmetric and $H_{\mathcal{B}}$ is a symplectic group.

Remark 1.1. The Coppersmith–Winograd tensor $T_{CW,q} \in \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2} \otimes \mathbb{C}^{q+2}$ is usually written as

$$T_{CW,q} = a_1 \otimes b_1 \otimes c_{q+2} + a_1 \otimes b_{q+2} \otimes c_1 + a_{q+2} \otimes b_1 \otimes c_1 + \sum_{\ell=2}^{q+1} \big(a_\ell \otimes b_\ell \otimes c_1 + a_\ell \otimes b_1 \otimes c_\ell + a_1 \otimes b_\ell \otimes c_\ell\big),$$

The expression for $T_{CW,m-2}$ in Theorem A is equivalent to the usual one, which can be seen by making the change of basis in C that permutes c_1 and c_{q+2}.

Remark 1.2. When $m \leq 14$ the classification problem is much more difficult. For example, when $m = 3$ there is a tensor with an $8 > 6 = \frac{3^2}{2} + \frac{3}{2}$ dimensional symmetry group, namely the unique up to scale skew-symmetric tensor.

Structure of the paper. In §2 we define the symmetry group of a tensor and describe how to compute its symmetry Lie algebra. In §3 we bound the dimensions of symmetry groups of 1_A-generic and binding tensors. The main sections of the paper are §4, §5, §6 and §7, where Theorem A is proved. In §8, we exhibit the symmetry Lie algebras of other tensors that have appeared in the study of the laser method. Central to Strassen's laser method is the *border rank* of the tensors employed to run it. In §9, we briefly discuss border ranks of tensors appearing in this work.

2. The symmetry group of a tensor

In this section, we define the symmetry group of a tensor and its Lie algebra.

Let $\widetilde{\Phi} : GL(A) \times GL(B) \times GL(C) \to GL(A \otimes B \otimes C)$ denote the natural action of $GL(A) \times GL(B) \times GL(C)$ on $A \otimes B \otimes C$. The map $\widetilde{\Phi}$ has a two dimensional kernel $\ker \widetilde{\Phi} = \{(\lambda \mathrm{Id}_A, \mu \mathrm{Id}_B, \nu \mathrm{Id}_C) : \lambda\mu\nu = 1\} \simeq (\mathbb{C}^*)^2$. Thus

$$G := (GL(A) \times GL(B) \times GL(C)) / (\mathbb{C}^*)^{\times 2} \tag{1}$$

is naturally a subgroup of $GL(A \otimes B \otimes C)$.

Definition 2.1. Let $T \in A \otimes B \otimes C$. The symmetry group of T, denoted G_T, is the stabilizer of T in G:

$$G_T := \{g \in (GL(A) \times GL(B) \times GL(C)) / (\mathbb{C}^*)^{\times 2} \mid \ g \cdot T = T\}. \tag{2}$$

The symmetry group G_T is an algebraic subgroup of $GL(A \otimes B \otimes C)$. We systematically compute $\dim G_T$ by determining the dimension of the corresponding Lie subalgebra, i.e., the annihilator of T in $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)) / \mathbb{C}^2$:

$$\mathfrak{g}_T = \{L \in (\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)) / \mathbb{C}^2 \mid L.T = 0\}.$$

The algebra $(\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)) / \mathbb{C}^2$ is the image of the differential $\Phi = d\widetilde{\Phi}$ of the map $\widetilde{\Phi}$ defined above (see e.g., [23, §1.2]).

It is more convenient to describe the annihilator $\widetilde{\mathfrak{g}}_T = \Phi^{-1}(\mathfrak{g}_T)$ as a subalgebra of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$, acting on $A \otimes B \otimes C$ via the Leibniz rule. Notice that $\widetilde{\mathfrak{g}}_T$ always contains $\ker \Phi = \{\lambda \mathrm{Id}_A, \mu \mathrm{Id}_A, \nu \mathrm{Id}_A) : \lambda + \mu + \nu = 0\} \simeq \mathbb{C}^2$, and so $\dim G_T = \dim \mathfrak{g}_T = \dim \widetilde{\mathfrak{g}}_T - 2$.

More explicitly, if $L = (U, V, W) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$, write $U = (u^i_j)$ as a matrix in the fixed bases, and similarly for V and W. The condition $L.T = 0$ is equivalent to the linear system of equations

$$u^i_{i'} T^{i'jk} + v^j_{j'} T^{ij'k} + w^k_{k'} T^{ijk'} = 0, \text{ for every } i, j, k. \tag{3}$$

Remark 2.2. We interpret equation (3) as follows: We view $u^i_{i'}, v^j_{j'}, w^k_{k'}$ as linear coordinates on $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$, i.e., as basis vectors of the dual space $\mathfrak{gl}(A)^* \oplus \mathfrak{gl}(B)^* \oplus \mathfrak{gl}(C)^*$. We have an inclusion $\widetilde{\mathfrak{g}}_T \subset \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ and (3) are the relations placed on these linear functions when they are pulled back to $\widetilde{\mathfrak{g}}_T$.

The codimension of $\widetilde{\mathfrak{g}}_T$ in $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$ equals the number of linearly independent equations in the system (3). In the rest of the paper, we often display special instances of (3), marking them with the corresponding triplet of indices (i, j, k).

If T and T' are isomorphic, then G_T and $G_{T'}$ are conjugate subgroups. We will use the action of G to *normalize* T; the normalizations will typically simplify the expression of T, allowing us to provide effective lower bounds on the rank of the linear system (3), which in turn are upper bounds on $\dim G_T$. It will be often useful to apply normalizations in several steps, with the understanding that subsequent normalizations must preserve the previous ones. When we describe our *freedom to normalize further*, we refer to possible normalizations that preserve the previous ones. For simplicity, we will always discuss normalizations by means of elements in $GL(A) \times GL(B) \times GL(C)$.

Semicontinuity of fiber dimension (see e.g., [24, I.6.3, Thm. 1.25]) implies that $\dim G_T$ is a lower semicontinuous function of T. In particular, for every s, the set $\{T \in A \otimes B \otimes C : \dim G_T \geq s\}$ is closed (its closures in the Zariski or Euclidean topologies coincide).

Remark 2.3. Let $T, T' \in A \otimes B \otimes C$. We say that T *degenerates* to T' if $T' \in \overline{G \cdot T}$. (Recall that G is defined in (1).) If T' is a degeneration of T, then $\dim G_{T'} \geq \dim G_T$. Moreover, if $T' \notin G \cdot T$ and T' is a degeneration of T, then $\overline{G \cdot T'} \subsetneq \overline{G \cdot T}$ and so $\dim G_{T'} > \dim G_T$.

Remark 2.3 implies :

Corollary 2.4. *When m is even $\overline{GL_m^{\times 3} \cdot T_{max,even,m}} \cap (1-generic\ tensors) = GL_m^{\times 3} \cdot T_{max,even,m}$. When m is odd $\overline{GL_m^{\times 3} \cdot T_{max,odd,skew,m}} \cap (1 - generic\ tensors) = GL_m^{\times 3} \cdot T_{max,odd,skew,m}$ and $\overline{GL_m^{\times 3} \cdot T_{CW,m-2}} \cap (1 - generic\ tensors) = GL_m^{\times 3} \cdot T_{CW,m-2}$.*

3. Symmetry groups of tensors: first results

In this section, we review the classical result on the largest possible symmetry group of any tensor, and we characterize the maximal possible symmetry group for a 1_A-generic tensor and for a binding tensor.

3.1. **Arbitrary tensors.** The unique tensor with largest symmetry group in $A\otimes B\otimes C$ is (up to change of bases) $a_1\otimes b_1\otimes c_1$. Its annihilator, presented in $(1,m-1)\times(1,m-1)$ block form, is

$$\widetilde{\mathfrak{g}}_{a_1\otimes b_1\otimes c_1}=\left\{\begin{pmatrix}u^1_1 & \mathbf{u}\\ 0 & \overline{U}\end{pmatrix},\begin{pmatrix}v^1_1 & \mathbf{v}\\ 0 & \overline{V}\end{pmatrix},\begin{pmatrix}w^1_1 & \mathbf{w}\\ 0 & \overline{W}\end{pmatrix}\ \Big|\ \begin{matrix}u^1_1+v^1_1+w^1_1=0,\\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{C}^{m-1},\overline{U},\overline{V},\overline{W}\in\mathfrak{gl}_{m-1}\end{matrix}\right\}.$$

Indeed, if $T=a_1\otimes b_1\otimes c_1$, the only equations of (3) that are not identically 0 are the ones with indices $(\rho 11),(1\rho 1)$ and (11ρ) and the one with indices (111). The first ones provide $u^\rho_1=v^\rho_1=w^\rho_1=0$ and the latter provides $u^1_1+v^1_1+w^1_1=0$.

Hence, $\dim G_{a_1\otimes b_1\otimes c_1}=[3(m-1)^2+3(m-1)+2]-2=3m^2-3m$. This can be obtained geometrically observing that the orbit of $a_1\otimes b_1\otimes c_1$ under the action of G is the cone over the Segre variety of rank one tensors, which has dimension $3m-2$.

Uniqueness follows from the fact that every tensor degenerates to $a_1\otimes b_1\otimes c_1$. In particular, if there are no $a\in A,b\in B,c\in C$ such that $T=a\otimes b\otimes c$, then $a_1\otimes b_1\otimes c_1\in\overline{G\cdot T}\setminus G\cdot T$ and therefore $\dim G_T<\dim G_{a_1\otimes b_1\otimes c_1}$ by Remark 2.3.

3.2. 1_A**-generic tensors.**

Proposition 3.1. *Let $T\in A\otimes B\otimes C$ be 1_A-generic. Then $\dim G_T\leq 2m^2-m-1$ and equality occurs uniquely for the tensor $T_0=a_1\otimes(\sum_{j=1}^m b_j\otimes c_j)$.*

Proof. Let $T\in A\otimes B\otimes C$ be 1_A-generic, so there exists $\alpha\in A^*$ such that $T_A(\alpha)\in B\otimes C$ has rank m. Use this to normalize T as follows: After a change of basis in B, we may assume that $T_A(\alpha)=\sum_{i=1}^m b_i\otimes c_i$, and by changing basis in A, we may further assume that $\alpha=\alpha^1$. That is, we may assume $T^{1jk}=\delta_{jk}$.

Applying (3) with $i=1$ gives

$$(1jk)\qquad u^1_1\delta_{jk}+u^1_\sigma T^{\sigma jk}+v^j_k+w^k_j=0. \tag{4}$$

Setting $i=\rho$ and $j=k$, (recall $2\leq\rho,\sigma,\tau\leq m$) one obtains

$$(\rho jj)\qquad u^\rho_1=-[u^\rho_\sigma T^{\sigma jj}+v^j_{j'}T^{\rho j'j}+w^j_{k'}T^{\rho jk'}]. \tag{5}$$

Now (4) shows that W is completely determined by u^1_i and V and (5) shows that u^ρ_1 is determined by u^ρ_σ,u^1_i and V. In summary, u^1_i, u^ρ_σ and V completely determine L. Thus $\dim\mathfrak{g}_T\leq(m+(m-1)^2+m^2)-2=2m^2-m-1$. Equality occurs when $T^{\rho jk}=0$ for every $\rho=2,\dots,m$ and every $j,k=1,\dots,m$. In this case $T=T_0=a_1\otimes(b_1\otimes c_1+\cdots+b_m\otimes c_m)$ and the relations above provide

$$\widetilde{\mathfrak{g}}_{T_0}=\left\{\begin{pmatrix}-(\mu+\nu) & \mathbf{u}\\ 0 & \overline{U}\end{pmatrix},(\mu\mathrm{Id}+V),(\nu\mathrm{Id}-V^{\mathbf{t}})\mid\mu,\nu\in\mathbb{C},\overline{U}\in\mathfrak{gl}_{m-1},V\in\mathfrak{gl}_m,\mathbf{u}\in\mathbb{C}^{m-1}\right\}.$$

To prove uniqueness, suppose that T is a 1_A-generic tensor and normalized as above. Then T degenerates to T_0 by applying the map $a_\rho\mapsto 0$ for $\rho=2,\dots,m$ on the space A.

If $\dim(T_A(A^*))=1$ then $T_A(A^*)=\langle T(\alpha^1)\rangle=\langle\sum_1^m b_i\otimes c_i\rangle$ and a change of basis in A sends T to T_0. If $\dim(T_A(A^*))\geq 2$, then $T_0\notin G\cdot T$ and therefore $\dim G_T<\dim G_{T_0}$ by Remark 2.3. $\square$

3.3. **Binding tensors.** Recall that T is binding implies there exist $\alpha \in A^*$ and $\beta \in B^*$ such that $T(\alpha) \in B \otimes C$ and $T(\beta) \in A \otimes C$ are full rank and thus induce identifications $B^* \simeq C$ and $A^* \simeq C$. Given a choice of such α, β we have a bilinear map $T : C \times C \to C$ such that $T(\alpha, \cdot) : C \to C$ and $T(\cdot, \beta) : C \to C$ are both the identity maps, and under our identifications, we view the bilinear map as inducing a (not necessarily associative) algebra with unit structure on C, with $\alpha \simeq \beta$ as the identity element.

Proposition 3.2. *Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be binding. Then $\dim G_T \leq m^2 - 1$, and equality occurs uniquely for the tensor*

$$T_{utriv,m} := a_1 \otimes b_1 \otimes c_1 + \sum_{\rho=2}^{m} a_1 \otimes b_\rho \otimes c_\rho + \sum_{\rho=2}^{m} a_\rho \otimes b_1 \otimes c_\rho.$$

Proof. Assume T is 1_A and 1_B generic. As in the proof of Proposition 3.1, we may assume $T(\alpha^1) \in B \otimes C$ has full rank and normalize it to $\sum b_i \otimes c_i$.

After the normalization $T^{1jk} = \delta_{jk}$, our freedom to normalize further is

$$\left\{ \left(\begin{pmatrix} x_0^0 & 0 \\ \mathbf{x} & \overline{X} \end{pmatrix}, \mu(Z^{\mathbf{t}})^{-1}, \nu Z \right) \mid \overline{X} \in GL_{m-1}, Z \in GL_m, x_0^0 \mu\nu = 1, \mathbf{x} \in \mathbb{C}^{m-1} \right\}, \tag{6}$$

the subgroup of $GL(A) \times GL(B) \times GL(C)$ preserving the condition $T^{1jk} = \delta_{jk}$. To see this, consider the set of $(X, Y, Z) \in GL(A) \times GL(B) \times GL(C)$ such that $\widetilde{T} = (X, Y, Z) \cdot T$ satisfies $\widetilde{T}^{1jk} = \delta_{jk}$ for every T satisfying $T^{1jk} = \delta_{jk}$. Then $\widetilde{T}^{1jk} = x^1_{i'} y^j_{j'} z^k_{k'} T^{i'j'k'}$ has to hold independently from the unassigned coefficients of T. This implies $x^1_\rho = 0$, and $x^1_1 y^j_{j'} z^k_{k'} T^{1j'k'} = \delta_{j'k'}$, which implies $Y = (Z^{\mathbf{t}})^{-1}$, up to scale.

Using the group in (6), we may assume $T(\beta^1) \in A \otimes C$ has full rank and normalize it to $T(\beta^1) = \sum_i a_i \otimes c_i$, that is $T^{i1k} = \delta_{ik}$.

After this normalization $T = T_{utriv,m} + T'$ where $T' \in \langle a_2, ..., a_m \rangle \otimes \langle b_2, ..., b_m \rangle \otimes C$. Apply the degeneration defined by $(X_\epsilon, Y_\epsilon, Z_\epsilon)$ with

$$X_\epsilon : \begin{array}{l} a_1 \mapsto \frac{1}{\epsilon} a_1 \\ a_\rho \mapsto \epsilon a_\rho \end{array} \qquad Y_\epsilon : \begin{array}{l} b_1 \mapsto \frac{1}{\epsilon} b_1 \\ b_\sigma \mapsto \epsilon b_\sigma \end{array} \qquad Z_\epsilon : \begin{array}{l} c_1 \mapsto \epsilon^2 c_1 \\ c_\tau \mapsto c_\tau. \end{array}$$

Among the bases elements appearing in T, notice that $a_i \otimes b_j \otimes c_k$ is fixed if and only if $(i, j, k) = (1, 1, 1)$ or $(i, j, k) = (1, \rho, \rho)$ or $(i, j, k) = (\rho, 1, \rho)$ and all the others have coefficient ϵ, ϵ^2, or ϵ^4. This shows that $\lim_{\epsilon \to 0} (X_\epsilon, Y_\epsilon, Z_\epsilon) \cdot T = T_{utriv,m}$, namely $T_{utriv,m} \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$. We conclude that either $T_{utriv,m} \in GL(A) \times GL(B) \times GL(C) \cdot T$, so T and $T_{utriv,m}$ are isomorphic, or $T_{utriv,m}$ is a proper degeneration of T, and therefore $\dim G_T < \dim G_{T_{utriv.m}}$ by Remark 2.3.

An explicit calculation gives

$$\widetilde{\mathfrak{g}}_{T_{utriv,m}} =$$
$$\left\{ \left(\begin{pmatrix} \lambda & \mathbf{u} \\ 0 & -(\mu+\nu)\mathrm{Id} - \overline{W}^{\mathbf{t}} \end{pmatrix}, \begin{pmatrix} \mu & \mathbf{v} \\ 0 & -(\lambda+\nu)\mathrm{Id} - \overline{W}^{\mathbf{t}} \end{pmatrix}, \begin{pmatrix} -\lambda-\mu & 0 \\ -\mathbf{u}^{\mathbf{t}} - \mathbf{v}^{\mathbf{t}} & \nu \mathrm{Id} + \overline{W} \end{pmatrix} \right) \mid \begin{array}{l} \lambda, \mu, \nu \in \mathbb{C}, \\ \mathbf{u}, \mathbf{v} \in \mathbb{C}^{m-1}, \overline{W} \in \mathfrak{sl}_{m-1} \end{array} \right\},$$

which has dimension $[(m-1)^2 - 1] + 2(m-1) + 3$. Hence $\dim \mathfrak{g}_{T_{utriv,m}} = \dim G_{T_{utriv,m}} = m^2 - 1$. This concludes the proof. □

Remark 3.3. Note that $T_{utriv,m}$ is concise. It has the largest dimensional symmetry group of any concise tensor we are aware of.

Problem 3.4. Determine the largest possible dimension of the symmetry group of a concise tensor. Furthermore, classify concise tensors with symmetry groups of maximal dimension.

4. Overview of proof of Theorem A

Since our tensor is binding, after choices of generic $\alpha \in A^*$ and $\beta \in B^*$, we obtain an identification $A \simeq B \simeq C^*$ and $\alpha \simeq \beta$. Choose a complement to α in A^* and adapt bases to the induced splitting, so we may write $A = L_1 \oplus M = \langle a_1 \rangle \oplus \langle a_2, ..., a_m \rangle$. 1-genericity implies the additional condition that there exists $\gamma \in C^*$ with $T(\gamma)$ full rank, which we may consider as a bilinear form on $A \otimes A$.

In the proof we show that in order to be in the range for Theorem A, the symmetry group of the bilinear form $T(\gamma)$ restricted to M must have large dimension, so to this aim, in §5, we determine the possible symmetry Lie algebras of non-degenerate bilinear forms on $\mathbb{C}^k \otimes \mathbb{C}^k$ with dimension at least $\binom{k}{2} - k$ (Lemma 5.1). There are seven cases, four with the dimension at least $\binom{k}{2}$, which we label **A1**,.., **A4**, and three of dimension less than $\binom{k}{2}$, labeled **B1**, **B2**,**B3**.

The proof of Theorem A splits into two cases: the generic one (treated in §6) where using the identification $A \simeq C^*$, we may take $\gamma = a_1$, and the non-generic where we may not (treated in §7).

In the generic case we may assume that our 1-generic tensor is of the form

$$T = a_1 \otimes a_1 \otimes c_1 + a_1 \otimes \mathrm{Id}_M + \sigma_{12}(a_1 \otimes \mathrm{Id}_M) + \mathcal{B} \otimes c_1 + \widehat{T} \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^{m*}, \tag{7}$$

where $\mathcal{B}|_{M \otimes M}$ is full rank, and $\widehat{T} \in M \otimes M \otimes M^*$, i.e., $\widehat{T} = T^{\rho\sigma\tau} a_\rho \otimes b_\sigma \otimes c_\tau$.

This normalization allows us to use (3) to obtain relations on the entries of $\widetilde{\mathfrak{g}}_T$ appearing in $L \otimes L \otimes L^*, M \otimes L \otimes L^*, L \otimes M \otimes L^*, L \otimes L \otimes M^*$ (given by equations (14)–(21)), and prove $\widetilde{\mathfrak{g}}_T$ can have dimension at most $(m-1)^2 + 3(m-1)$, where $(m-1)^2 = \dim \mathfrak{gl}(M)$, see (22). We next show that the component of $\widetilde{\mathfrak{g}}_T$ in $\mathfrak{gl}(M)$ must annihilate the bilinear form $\mathcal{B}$, see (23). In particular, to be eligible for consideration in Theorem A, this means that the Lie algebra annihilating $\mathcal{B}$, denoted $\mathfrak{h}_\mathcal{B} \subset \mathfrak{gl}(M)$, must have dimension at least $\binom{m-1}{2} - (m-1)$. We then apply Lemma 5.1 to normalize $\mathcal{B}$ to one of the seven cases of the Lemma. At this point, the remaining unknown coefficients of T are those in $\widehat{T}$, namely $T^{\rho\sigma\tau}$. The bulk of the work goes into showing these coefficients must be zero. After that, it is an easy calculation to see that only case **A1** of Lemma 5.1 appears in the theorem in this generic case.

To show the coefficients $T^{\rho\sigma\tau}$ are zero, one takes advantage of the fact that $M \otimes M \otimes M^*$ is an $\mathfrak{h}_\mathcal{B}$-module, and that $\mathfrak{h}_\mathcal{B}$ must annihilate $\widehat{T}$. Thanks to Schur's lemma, this annihilator must annihilate the components of $\widehat{T}$ in each $\mathfrak{h}_\mathcal{B}$-irreducible submodule of $M \otimes M \otimes M^*$. There are three distinguished submodules, two isomorphic to M and one to M^*, embedded as $M \otimes \mathrm{Id}_M$, $\sigma_{12}(M \otimes \mathrm{Id}_M)$ and $\mathcal{B} \otimes M^*$. Write $(M \otimes M \otimes M)_{prim}$ for the sum of the remaining components. Then it is easy to see (Lemma 6.2), if there is not a subspace $R \subset M$ of dimension at most two, such that $\widehat{T}$ takes values in $R \otimes \mathrm{Id}_M$, $\sigma_{12}(R \otimes \mathrm{Id}_M)$, $\mathcal{B} \otimes R^*$, and $(R \otimes R \otimes R)_{prim}$, it cuts $\dim \mathfrak{h}_\mathcal{B}$ down too much for consideration in the theorem. We then eliminate the cases $\dim R = 1$ and $\dim R = 2$ separately in Lemmas 6.3 and 6.4. The computation in the case $\dim R = 1$ is long.

In the non-generic case, where $T(a_1)$ drops rank (again considering $a_1 \in C^*$), we choose a further splitting of M to $N \oplus L_m$ where $\dim L_m = 1$, and $T|_{(N \otimes N \otimes L_m)^*}$ is nondegenerate (in the proof L_m is spanned by c_m). Our space now has $27 = 3^3$ components that must be analyzed. Fifteen

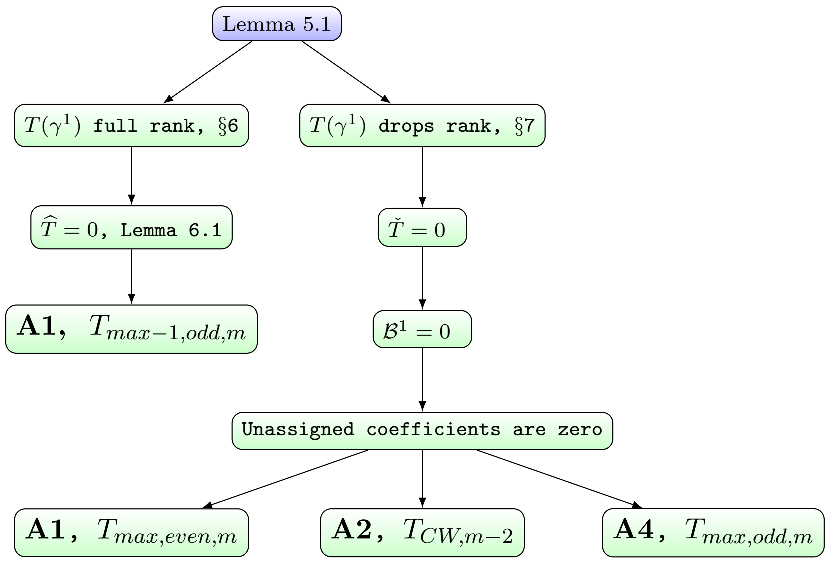

FIGURE 1. The flow diagram of the results leading to the proof of Theorem A.

of the components are easy and the $N \otimes N \otimes L_m$ component (after a choice of basis vector for L_m) by hypothesis is a nondegenerate bilinear form $\mathcal{B} \in N \otimes N$ (denoted $T^{stm} a_s \otimes b_t$ in the proof), and again we let $\mathfrak{h}_{\mathcal{B}} \subset N \otimes N^*$ denote its annihilator. A dimension count (see the inequality (59)) similar to the generic case reduces the possible bilinear forms to cases **A1**-**A4** of Lemma 5.1. Analogously with the previous case, to appear in Theorem A, T cannot have any support in the $N \otimes N \otimes N^*$ component (i.e., $\check{T} = 0$ in the notation of the proof). We then study the $N \otimes N \otimes L_1$ component, which determines a bilinear form that we have to show is zero (denoted $\mathcal{B}^1$ in the proof). We finally show that for each of the remaining components, if T had any nonzero support in any of them, it would be removed from consideration in the theorem. Finally we compute $\mathfrak{g}_T$ for the four (explicit) remaining cases. In 3 of the 4 cases, the tensor is eligible for Theorem A, but in the fourth, the "nilpotent" part of the stabilizer is too small.

The flow diagram of the proof is depicted in Figure 1.

Remark 4.1. The proof of Theorem A proceeds by calculations inspired by the Exterior Differential Systems pioneered by E. Cartan (see, e.g., the classic [10]) and modernized by R. Bryant (see, e.g., [7]).

5. Symmetry algebras of bilinear forms

This section deals with the symmetry group of a non-degenerate bilinear form, that is the stabilizer of a full rank element $\mathcal{B} \in \mathbb{C}^k \otimes \mathbb{C}^k$ under the action of GL_k given by $g \cdot \mathcal{B} = g\mathcal{B}g^{\mathbf{t}}$. Let $H_{\mathcal{B}}$ be this stabilizer and let $\mathfrak{h}_{\mathcal{B}}$ be its Lie algebra. An element $X \in \mathfrak{h}_{\mathcal{B}}$ is characterized by the condition

$$X\mathcal{B} + \mathcal{B}X^{\mathbf{t}} = 0. \tag{8}$$

Let $W = \mathbb{C}^{k*}$. As a $\mathfrak{gl}(W)$-module, $W^* \otimes W^* = S^2W^* \oplus \Lambda^2 W^*$. Write $\mathcal{B} = Q + \Lambda$ with $Q \in S^2W^*$ symmetric and $\Lambda \in \Lambda^2 W^*$ skew-symmetric. Write $E = \ker(\Lambda)$, $F = \ker(Q)$, which are subspaces of W; let $L^* = E^\perp \cap F^\perp \subseteq W^*$. Choose a complement $L \subset W$ of $E \oplus F$ so that we have a direct sum decomposition

$$W = E \oplus L \oplus F,$$

and may identify L with the dual space of L^*. We also may identify $E^* = (L \oplus F)^\perp \subset W^*$ and $F^* = (E \oplus L)^\perp \subset W^*$, where normally these spaces would respectively be defined as $W^*/E^\perp$ and $W^*/F^\perp$.

Let $e = \dim E$, $f = \dim F$, and $\ell = \dim L$. Notice that $\mathrm{rk}(\Lambda) = \ell + f$ is even.

Adopt the following notation for a subspace $U \subset W$, write $\mathcal{B}|_U := \mathcal{B}|_{U\times U}$ which is an element of $(W^*/U^\perp) \otimes (W^*/U^\perp) = U^* \otimes U^*$.

Lemma 5.1. *With notations as above, let $k \geq 12$ and let $\mathcal{B} \in \mathbb{C}^k \otimes \mathbb{C}^k$ be a full rank bilinear form. Then*

$$\dim \mathfrak{h}_{\mathcal{B}} \leq \binom{k}{2} - k - 1 = \frac{k^2}{2} - \frac{3k}{2} - 1,$$

except for the following cases:

- **A1**. $(e, \ell, f) = (0, 0, k)$ *(so k is even): in this case $\mathcal{B} = \Lambda$ is skew-symmetric and $\mathfrak{h}_{\mathcal{B}} = \mathfrak{sp}(\Lambda)$ with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k+1}{2}$;
- **A2**. $(e, \ell, f) = (k, 0, 0)$*: in this case $\mathcal{B} = Q$ is symmetric and $\mathfrak{h}_{\mathcal{B}} = \mathfrak{so}(Q)$ with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2}$;
- **A3**. $(e, \ell, f) = (0, 1, k-1)$ *(so k is even): in this case* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} + 1$;
- **A4**. $(e, \ell, f) = (1, 0, k-1)$ *(so k is odd): $\mathfrak{h}_{\mathcal{B}} = \mathfrak{sp}(\Lambda|_F)$ with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2}$;
- **B1**. $(e, \ell, f) = (0, 2, k-2)$ *(so k is even): there are two cases, described in the proof, with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} - k + 3$ *or* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} - k + 2$;
- **B2**. $(e, \ell, f) = (1, 1, k-2)$ *(so k is odd): there are two cases, described in the proof, with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} - k + 1$ *in both cases;*
- **B3**. $(e, \ell, f) = (2, 0, k-2)$ *(so k is even) $\mathfrak{h}_{\mathcal{B}} = \mathfrak{so}(Q|_E) \oplus \mathfrak{sp}(\Lambda|_F)$, with* $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} - k + 1$.

Remark 5.2. While there is extensive literature on orbits in the adjoint representations of $\mathfrak{so}_k, \mathfrak{sp}_k$, e.g., [30, 9, 16, 20], we were unable to find any references where the skew and symmetric forms were allowed to be degenerate with their sum nondegenerate.

Proof. The condition $X\mathcal{B} + \mathcal{B}X^{\mathbf{t}} = 0$ is equivalent to the two conditions

$$XQ + QX^{\mathbf{t}} = 0,$$
$$X\Lambda + \Lambda X^{\mathbf{t}} = 0.$$

The solution space of $XQ + QX^{\mathbf{t}} = 0$ is the Lie algebra $\mathfrak{so}(Q|_{E\oplus L}) \rtimes F^* \otimes (E \oplus L)$ and the solution space of $X\Lambda + \Lambda X^{\mathbf{t}} = 0$ is the Lie algebra $\mathfrak{sp}(\Lambda|_{L\oplus F}) \rtimes E^* \otimes (L \oplus F)$; therefore we have

$$\mathfrak{h}_{\mathcal{B}} = [\mathfrak{so}(Q|_{E\oplus L}) \rtimes F^* \otimes (E \oplus L)] \cap [\mathfrak{sp}(\Lambda|_{L\oplus F}) \rtimes E^* \otimes (L \oplus F)] \subseteq W \otimes W^*.$$

The proof deals with a number of special cases for small values of e, ℓ and f and provides a general argument when $e \geq 2$ and $\ell, f > 0$.

First, we consider the following special cases:

- Case $e = 0$. In this case Λ has full rank and $k = \ell + f$ is even. In this case, $\mathfrak{h}_{\mathcal{B}}$ is the annihilator of $Q \in S^2 W^*$ in $\mathfrak{sp}(\Lambda)$; its codimension in $\mathfrak{sp}(\Lambda)$ is the dimension of the $SP(\Lambda)$-orbit of Q.
  - Subcase $\ell = 0$. In this case $f = k$ and $Q = 0$. We have $\mathcal{B} = \Lambda$ and $\mathfrak{h}_{\mathcal{B}} = \mathfrak{sp}(\Lambda)$ with $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k+1}{2}$. This is case **A1**.
  - Subcase $\ell = 1$. In this case $f = k-1$ and $\mathrm{rk}(Q) = 1$. Therefore $\mathfrak{h}_{\mathcal{B}}$ is the annihilator of Q in $\mathfrak{sp}(\Lambda)$. Rank one elements in S^2W^* are equivalent under the action of $SP(\Lambda)$: the $SP(\Lambda)$-orbit of $[Q]$ is the Veronese variety $\nu_2(\mathbb{P}W^*)$, which has dimension $k-1$. Therefore $\dim \mathfrak{h}_{\mathcal{B}} = \dim \mathfrak{sp}(\Lambda) - (k-1) = \binom{k+1}{2} - k + 1 = \binom{k}{2} + 1$. This is case **A3**.
  - Subcase $\ell = 2$. In this case $f = k-2$ and $\mathrm{rk}(Q) = 2$. Therefore $\mathfrak{h}_{\mathcal{B}}$ is the annihilator of Q in $\mathfrak{sp}(\Lambda)$. The group $SP(\Lambda)$ has two orbits in the set of rank two elements in S^2W^*: there are elements $w_1 + w_2$, with $[w_j] \in \nu_2(\mathbb{P}W)$ such that $\Lambda(w_1, w_2) = 0$ or with $\Lambda(w_1, w_2) \neq 0$. In the first case, the orbit-closure of $[Q]$ is the set of points lying on a *contact* tangent line to $\nu_2(\mathbb{P}W))$ (see e.g., [20, §6] for an extensive discussion on the contact structure induced on $\nu_2(\mathbb{P}W)$ by the Lie algebra $\mathfrak{sp}(\Lambda)$); this has dimension $2k-3$, providing $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k+1}{2} - (2k-3)$. In the second case, the orbit-closure of $[Q]$ is the tangential variety of $\nu_2(\mathbb{P}W^*)$, which has dimension $2k-2$, providing $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k+1}{2} - (2k-2)$. This is case **B1**.
  - Subcase $\ell = 3$. In this case $f = k-3$ and $\mathrm{rk}(Q) = 3$. Therefore $H_{\mathcal{B}}$ is the stabilizer of Q in $SP(\Lambda)$. In particular, it has to stabilize $(\ker(Q))^{\perp} = F^{\perp} = L^*$, as an element of the Grassmannian $G(3, W^*) \subseteq \mathbb{P}\Lambda^3 W^*$. This shows $H_{\mathcal{B}} \subseteq P_{L^*}$, where P_{L^*} is the stabilizer of L^*. The only closed $SP(\Lambda)$-orbit in $\mathbb{P}\Lambda^3 W^*$ is the Lagrangian Grassmannian of isotropic 3-planes $G_\Lambda(3, W^*)$, with $\dim G_\Lambda(3, W^*) = 3k - 12$. In particular $\dim P_{L^*} \leq \dim \mathfrak{sp}(\Lambda) - \dim G_\Lambda(3, W^*)$. By semicontinuity, we conclude $\dim H_{\mathcal{B}} \leq \dim P_{L^*} \leq \mathfrak{sp}(\Lambda) - (3k-12) = \frac{k^2}{2} + \frac{k-6k+24}{2} = \frac{k^2}{2} - \frac{5k}{2} + 12$ which is less than $\frac{k^2}{2} - \frac{3k}{2} - 1$ when $k \geq 12$. (A direct calculation with a variant of Terracini's lemma shows $\dim H_{\mathcal{B}} = \frac{k^2}{2} - \frac{5k}{2} + 12$.)
  - Subcase $\ell \geq 4$. In this case, normalize Q to be $Q = v_1^2 + \cdots + v_\ell^2$ where $v_1, \ldots, v_k$ is a basis of W^*. Consider $Q_\epsilon = v_1^2 + \cdots + v_3^2 + \epsilon(v_4^2 + \cdots + v_\ell^2)$. It is clear that the dimension of the annihilator of Q_ϵ in $\mathfrak{sp}(\Lambda)$ is constant for $\epsilon \neq 0$. By semicontinuity, we deduce that $\mathfrak{h}_{\mathcal{B}}$ has dimension less than the dimension of the annihilator of Q_0, that is the dimension that we determined in the previous case ($\ell = 3$). In particular, we have $\dim \mathfrak{h}_{\mathcal{B}} \leq \frac{k^2}{2} - \frac{3k}{2} - 1$.
- Case $f = 0$. In this case Q has full rank and ℓ is even. The algebra $\mathfrak{h}_{\mathcal{B}}$ is the annihilator of $\Lambda \in \Lambda^2 W^*$ in $\mathfrak{so}(Q)$. Its codimension in $\mathfrak{so}(Q)$ is the dimension of the $SO(Q)$-orbit of Λ.
  - Subcase $\ell = 0$. In this case $e = k$ so $\Lambda = 0$ and $\mathcal{B} = Q$. Therefore $\mathfrak{h}_{\mathcal{B}} = \mathfrak{so}(Q)$, with $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2}$. This is case **A2**.
  - Subcase $\ell \geq 2$. In this case $\mathrm{rk}(\Lambda) = 2$. The unique closed $SO(Q)$-orbit in $\mathbb{P}\Lambda^2 W^*$ is the Grassmannian $G_Q(2, W^*)$ of isotropic 2-planes in W^*, which has dimension $\dim G_Q(2, W^*) = 2(k-4) + 1$. By semicontinuity, we deduce $\dim \mathfrak{h}_{\mathcal{B}} \leq \dim \mathfrak{so}(Q) -$

$\dim G_Q(2,W^*) = \binom{k}{2} - 2(k-4)+1 = \frac{k^2}{2} - \frac{5k}{2} - 7$, which is less than $\frac{k^2}{2} - \frac{3k}{2} - 1$ when $k \geq 8$.

   – Subcase $\ell \geq 4$. An argument similar to the one of case $e = 0, \ell \geq 4$ applies and we conclude, again, by semicontinuity.

- Case $\ell = 0$, $e, f > 0$. In this case $\mathfrak{h}_{\mathcal{B}} = \mathfrak{so}(Q|_E) \oplus \mathfrak{sp}(\Lambda|_F)$ and therefore $\dim \mathfrak{h}_{\mathcal{B}} = \binom{e}{2} + \binom{f+1}{2}$.
   – Subcase $e = 1$. We have $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2}$. This is case **A4**.
   – Subcase $e = 2$. We have $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k-1}{2} + 1 = \binom{k}{2} - k + 2$. This is case **B3**.
   – Subcase $3 \leq e \leq k-1$. The value $\dim \mathfrak{h}_{\mathcal{B}} = \binom{e}{2} + \binom{(k-e)+1}{2}$ is a convex function of e; since for $e = 3$ and $e = k-1$ we obtain $\dim \mathfrak{h}_{\mathcal{B}} \leq \frac{k^2}{2} - \frac{3k}{2} + 1$, we conclude.
- Case $e = 1$ and $\ell, f > 0$. In this case k is odd and $\mathrm{rk}(\Lambda) = k-1$. By choosing a basis according to the splitting $E \oplus (L \oplus F)$, we have

$$\mathfrak{h}_\Lambda := \mathfrak{sp}(\Lambda) \rtimes [E^* \otimes (L \oplus F)] = \left\{ \begin{pmatrix} x_1^1 & \mathbf{x} \\ 0 & \overline{X} \end{pmatrix} : \overline{X} \in \mathfrak{sp}(\Lambda|_{L \oplus F}), \mathbf{x} \in \mathbb{C}^{k-1} \right\}, \tag{9}$$

which has dimension $\binom{k}{2} + k$.

   – Subcase $\ell = 1$, so $\mathrm{rk}(Q) = 2$. Then we have two cases, depending on whether $Q|_E$ is zero or not. In either case, we may normalize Q to be zero outside the upper left 2×2 submatrix; moreover, if $Q|_E \neq 0$, normalize $Q|_E$ to be Id_2 and if $Q|_E = 0$, normalize it to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then a direct calculation gives $\dim \mathfrak{h}_{\mathcal{B}} = \binom{k}{2} - k + 1$ in both cases. This is case **B2**.
   – Subcase $\ell = 2$, so $\mathrm{rk}(Q) = 3$. Again, there are two cases to consider, depending on whether E is isotropic for Q or not. An explicit calculation similar to the one of the previous case shows that in both cases we obtain $\dim \mathfrak{h}_{\mathcal{B}} < \binom{k}{2} - k - 1$.
   – Subcase $\ell \geq 3$. As in the previous cases, first we consider the case $\ell = 3$, using that $\mathfrak{h}_{\mathcal{B}}$ must be contained in the parabolic subalgebra determined by $L^* \subseteq E^\perp$. This gives an upper bound $\dim \mathfrak{h}_{\mathcal{B}} \leq \binom{k}{2} - 2k + 9$. If $\ell > 3$, we conclude via a semicontinuity argument as before.

The rest of the proof is obtained with a uniform argument with $e \geq 2$, $\ell, f > 0$. Up to redefining L, the symmetric form Q can be normalized so that

$$Q|_{E \oplus L} = \begin{pmatrix} \mathrm{Id}_q & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{e-q} & 0 \\ 0 & \mathrm{Id}_{e-q} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{\ell-e+q} \end{pmatrix} \tag{10}$$

where the blocking is $(q, e-q, e-q, \ell-e+q)$ with $q = \mathrm{rk}(Q|_E)$. To do this, define $E_2 = \ker(Q|_E)$, which by definition is isotropic and $\dim E_2 = e - q$. Let L_1 be an isotropic subspace of W such that $Q|_{E_2 \times L_1}$ is non-degenerate: in particular, L_1 is disjoint from $E \oplus F$ since $Q|_{E_2 \times (E \oplus F)}$ is identically 0. Now, the left kernel K of $Q|_{W \times (E \oplus L_1)}$ contains F and is disjoint from E and L_1; let L_2 be a complement of F in K; notice that $Q|_{L_2}$ is nondegenerate. Let $L = L_1 \oplus L_2$. Finally, consider $E_1 \subseteq E$ such that $Q|_{E_1 \times (E_2 \oplus L)} = 0$: notice that $Q|_{E_1}$ is non-degenerate. The splitting $E \oplus L = E_1 \oplus E_2 \oplus L_1 \oplus L_2$ provides (after choosing basis), the representation of (10).

Write X in block form (according to the decomposition $W = E \oplus L \oplus F$) as:

$$X = \begin{bmatrix} X_{EE} & X_{EL} & X_{EF} \\ X_{LE} & X_{LL} & X_{LF} \\ X_{FE} & X_{FL} & X_{FF} \end{bmatrix}.$$

The equation $XQ + QX^{\mathbf{t}} = 0$ implies $\begin{pmatrix} X_{EE} & X_{EL} \\ X_{LE} & X_{LL} \end{pmatrix} \in \mathfrak{so}(Q)$ and $\begin{pmatrix} X_{FE} & X_{FL} \end{pmatrix} = 0$.

Similarly, the equation $X\Lambda + \Lambda X^{\mathbf{t}} = 0$ implies $\begin{pmatrix} X_{LL} & X_{LF} \\ X_{FL} & X_{FF} \end{pmatrix} \in \mathfrak{sp}(L \oplus F, \Lambda)$ and $\begin{pmatrix} X_{EL} & X_{EF} \end{pmatrix} = 0$.

In summary, we have

$$X = \begin{pmatrix} X_{EE} & 0 & 0 \\ X_{LE} & X_{LL} & X_{LF} \\ 0 & 0 & X_{FF} \end{pmatrix} \quad \text{with} \quad \begin{array}{l} \begin{pmatrix} X_{EE} & 0 \\ X_{LE} & X_{LL} \end{pmatrix} \in \mathfrak{so}_{e+\ell}(Q), \\ \\ \begin{pmatrix} X_{LL} & X_{LF} \\ 0 & X_{FF} \end{pmatrix} \in \mathfrak{sp}_{\ell+f}(\Lambda). \end{array} \tag{11}$$

Now consider the upper right block of size $(e + \ell)$, and write it according to the $(q, e - q, e - q, \ell - (e - q))$ described above

$$\begin{pmatrix} X_{EE} & 0 \\ X_{LE} & X_{LL} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}$$

Then $XQ + QX^{\mathbf{t}} = 0$ implies

$$\begin{pmatrix} X_{EE} & 0 \\ X_{LE} & X_{LL} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & X_{22} & 0 & 0 \\ -X_{12}^{\mathbf{t}} & X_{32} & -X_{22}^{\mathbf{t}} & X_{34} \\ 0 & -X_{34}^{\mathbf{t}} & 0 & X_{44} \end{pmatrix} \quad \text{with} \quad \begin{array}{l} X_{11} \in \mathfrak{so}_q, \\ X_{44} \in \mathfrak{so}_{\ell-e+q}, \\ X_{32} \in \mathfrak{so}_{e-q}, \end{array}$$

with $X_{12} \in \mathbb{C}^{e-q} \otimes \mathbb{C}^q$, $X_{34} \in \mathbb{C}^{\ell-e+q} \otimes \mathbb{C}^{e-q}$, $X_{22} \in \mathbb{C}^{e-q} \times \mathbb{C}^{e-q}$.

Thus the total contribution of this block to the dimension of $\mathfrak{h}_{\mathcal{B}}$ will be bounded by

$$q(e-q) + (e-q)(e-q) + (e-q)(\ell - (e-q)) + \binom{q}{2} + \binom{\ell+q-e}{2} + \binom{e-q}{2} = \binom{q}{2} + \binom{\ell}{2} - eq + e^2.$$

Now, write Λ according to the decomposition $L \oplus F$, so that

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ -\Lambda_{12}^{\mathbf{t}} & \Lambda_{22} \end{pmatrix},$$

where Λ_{11} and Λ_{22} are skew-symmetric.

Let $\kappa = f - \mathrm{rk}(\Lambda_{22})$; notice that κ has the same parity as f since Λ_{22} is skew-symmetric. Via the action of $SL(F)$, we may normalize Λ_{22} to $\Lambda_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\Lambda}_{22} \end{pmatrix}$, in block form $(\kappa, f - \kappa)$ where $\widetilde{\Lambda}_{22}$ is a non-degenerate skew-symmetric matrix of size $f - \kappa$. After normalizing Λ_{22}, we see that the first κ columns of Λ_{12} are linearly independent, because $\Lambda|_{L\oplus F}$ has full rank: therefore via the action of $SL(L)$, we may normalize Λ_{12} to $\Lambda_{12} = \left(\begin{smallmatrix} P & P_1 \\ 0 & P_2 \end{smallmatrix}\right)$ in block form $(\kappa, \ell-\kappa) \times (\kappa, f-\kappa)$, for some matrices P, P_1, P_2, with P invertible of size κ. In particular, this shows $\kappa \leq \min\{f, \ell\}$.

The condition $X\Lambda + \Lambda X^{\mathbf{t}} = 0$ provides the following three equations (where we use that $X_{FL} = 0$):

$$\begin{aligned} &X_{LL}\Lambda_{11} - X_{LF}\Lambda_{12}^{\mathbf{t}} + \Lambda_{11}X_{LL}^{\mathbf{t}} + \Lambda_{12}X_{LF}^{\mathbf{t}} = 0 \\ &X_{LL}\Lambda_{12} + X_{LF}\Lambda_{22} + \Lambda_{12}X_{FF}^{\mathbf{t}} = 0 \\ &X_{FF}\Lambda_{22} + \Lambda_{22}X_{FF} = 0. \end{aligned} \tag{12}$$

The last relation, implies that $X_{FF} \in \mathfrak{h}_{\Lambda_{22}} := \mathfrak{sp}(\widetilde{\Lambda}_{22}) \rtimes [\mathbb{C}^f \otimes \mathbb{C}^\kappa]$.

From the first and second equations, we deduce $\Lambda_{12}X_{LF}^{\mathbf{t}} - X_{LF}\Lambda_{12}^{\mathbf{t}} \equiv 0 \mod \{X_{LL}, X_{FF}\}$ and $X_{LF}\Lambda_{22} \equiv 0 \mod \{X_{LL}, X_{FF}\}$. In order to determine how many linearly independent relations on X_{LF} this provides, write X_{LF} in block form $(\kappa, \ell-\kappa) \times (\kappa, f-\kappa)$ as $X_{LF} = \left(\begin{smallmatrix} X_{LF,11} & X_{LF,12} \\ X_{LF,21} & X_{LF,22} \end{smallmatrix}\right)$. The condition $X_{LF}\Lambda_{22} = 0$ provides $X_{LF,12} = 0$ and $X_{LF,22} = 0$ because $\widetilde{\Lambda}_{22}$ has full rank. Finally, the condition $\Lambda_{12}X_{LF}^{\mathbf{t}} - X_{LF}\Lambda_{12}^{\mathbf{t}} = 0$ gives $X_{LF,21} = 0$ and $X_{LF,11}^{\mathbf{t}}P - PX_{LF,11} = 0$, which provides $\binom{\kappa}{2}$ conditions since P has full rank.

In summary, we conclude

$$\dim \mathfrak{h}_{\mathcal{B}} \leq \underbrace{\binom{q}{2} + \binom{\ell}{2} - eq + e^2}_{X_{EE}, X_{LL}, X_{EL}} + \underbrace{\binom{f-\kappa+1}{2} + f\cdot\kappa}_{X_{FF}} + \underbrace{\binom{\kappa+1}{2}}_{X_{LF}} \tag{13}$$

We will conclude by showing that the right hand side of the inequality above is smaller than $\binom{k}{2} - k - 1$. Consider the difference:

$$\begin{aligned} \left[\binom{k}{2} - k - 1\right] - \dim \mathfrak{h}_{\mathcal{B}} \geq \left[\binom{k}{2} - k - 1\right] - \left[\binom{q}{2} + \binom{\ell}{2} - eq + e^2 + \binom{f-\kappa+1}{2} + f\cdot\kappa + \binom{\kappa+1}{2}\right] = \\ qe + \ell e - \tfrac{1}{2}q^2 - \tfrac{1}{2}e^2 + \ell f - \kappa^2 + ef - 2f - \ell - \tfrac{1}{2}q - \tfrac{3}{2}e - 1 = \\ \underbrace{\left[e(q+\ell) - \tfrac{1}{2}q^2 - \tfrac{1}{2}e^2\right]}_{=:S_1} + \underbrace{\left[\ell f - \kappa^2\right]}_{=:S_2} + \underbrace{\left[ef - 2f\right]}_{=:S_3} + \underbrace{\left[-\ell - \tfrac{1}{2}q - \tfrac{3}{2}e - 1\right]}_{=:S_4}. \end{aligned}$$

Recall that $\kappa \leq \ell, f$ and that $q \leq e$ and $e - q \leq \ell$. This implies

$$\begin{aligned} S_1 &= e(q+\ell) - \frac{1}{2}e^2 - \frac{1}{2}q^2 \geq 0; \\ S_2 &= \ell f - \kappa^2 \geq 0; \\ S_3 &= ef - 2f \geq 0. \end{aligned}$$

Suppose $q + \ell = e$, Then $S_1 + S_4 = q(\ell-2) + \frac{1}{2}\ell(\ell-5) - 1$. If $\ell \geq 5$, we conclude. If $2 \leq \ell \leq 4$, then $\frac{1}{2}\ell(\ell-5) \geq -3$: if $f > 5$, then we obtain a lower bound on S_2 using $\kappa \leq \ell, f$, which guarantees $S_1 + S_2 + S_4 \geq 0$; if $f \leq 5$, then $e \geq k - \ell - 5$ and we obtain a lower bound on S_3 which guarantees $S_1 + S_3 + S_4 \geq 0$. If $\ell = 1$, then $S_1 + S_4$ reduces to $-(q+3)$: if $f \geq 2$ then $S_1 + S_3 + S_4 \geq 0$ because $e = q + 1$. If $f = 1$, then we are in the case $(e, \ell, f) = (e, 1, 1)$ with $q = e - 1$, which can be analyzed explicitly obtaining that $\mathfrak{h}_{\mathcal{B}}$ is smaller than the desired bound.

The analysis when $q+\ell>e$ is similar, and can be done introducing a variable $s\geq 0$ such that $q+\ell=e+1+s$. □

6. Proof of Theorem A in the case $T(\gamma^1)$ is full rank

We have the normalizations $T^{1jk}=\delta_{jk}$ and $T^{i1k}=\delta_{ik}$ and the splittings and identifications $A\simeq B\simeq C^*\simeq L_1\oplus M$, where $L_1\simeq\langle a_1\rangle$, $M\simeq\langle a_2,...,a_m\rangle$. We rewrite (3) for the 8 possible choices of types of indices.

(14) (111) $u^1_1+v^1_1+w^1_1=0,$

(15) $(\rho 11)$ $u^\rho_1+v^1_\sigma T^{\rho\sigma 1}+w^1_\rho=0,$

(16) $(1\sigma 1)$ $u^1_\rho T^{\rho\sigma 1}+v^\sigma_1+w^1_\sigma=0,$

(17) (11τ) $u^1_\tau+v^1_\tau+w^\tau_1=0,$

(18) $(1\sigma\tau)$ $u^1_1\delta_{\sigma\tau}+u^1_\rho T^{\rho\sigma\tau}+v^\sigma_\tau+w^\tau_\sigma=0,$

(19) $(\rho 1\tau)$ $u^\rho_\tau+v^1_1\delta_{\rho\tau}+v^1_\sigma T^{\rho\sigma\tau}+w^\tau_\rho=0,$

(20) $(\rho\sigma 1)$ $u^\rho_{\rho'}T^{\rho'\sigma 1}+v^\sigma_{\sigma'}T^{\rho\sigma' 1}+w^1_1 T^{\rho\sigma 1}+w^1_\tau T^{\rho\sigma\tau}=0,$

(21) $(\rho\sigma\tau)$ $u^\rho_1\delta_{\sigma\tau}+v^\sigma_1\delta_{\rho\tau}+w^\tau_1 T^{\rho\sigma 1}+u^\rho_{\rho'}T^{\rho'\sigma\tau}+v^\sigma_{\sigma'}T^{\rho\sigma'\tau}+w^\tau_{\tau'}T^{\rho\sigma\tau'}=0.$

We remind the reader (see Remark 2.2) that we interpret these equations as relations imposed among the basis vectors of $(\mathfrak{gl}(A)\oplus\mathfrak{gl}(B)\oplus\mathfrak{gl}(C))^*$ when they are pulled back to $\widetilde{\mathfrak{g}}_T$.

Let

$$I:=\langle u^1_\rho, v^1_\sigma, w^1_\tau, u^1_1, v^1_1\rangle,$$

and note that $\dim I=3(m-1)+2$.

First we show that under the conditions (14)–(21), the entries of (U,V,W) pulled back to $\widetilde{\mathfrak{g}}_T$ are completely determined by I and $\overline{U}:=(u^\rho_\sigma)$. Equation (14) implies $w^1_1\equiv 0 \mod I$, that is w^1_1 pulled back to $\widetilde{\mathfrak{g}}_T$ is a linear combination of elements of I pulled back to $\widetilde{\mathfrak{g}}_T$. Moreover, from (15) $u^\rho_1\equiv 0 \mod I$, from (16) $v^\sigma_1\equiv 0 \mod I$ and from (17) $w^\tau_1\equiv 0 \mod I$. Finally, from (18) and (19), one has $\overline{U}\equiv -\overline{W}^{\mathbf{t}}\equiv\overline{V} \mod I$, and therefore $\overline{V}:=(v^\rho_\sigma),\overline{W}:=(w^\rho_\sigma)$ are uniquely determined by the entries of $\overline{U}$ and I.

At this point we have

$$\dim\widetilde{\mathfrak{g}}_T\leq 3(m-1)+2+(m-1)^2 \tag{22}$$

as all entries in $\widetilde{\mathfrak{g}}_T$ may be expressed as linear combinations of elements of I and $\overline{U}$ pulled back to $\widetilde{\mathfrak{g}}_T$.

Now, using $\overline{U}\equiv\overline{V} \mod I$ to substitute out $\overline{V}$ from (20), we obtain

$$u^\rho_{\rho'}T^{\rho'\sigma 1}+u^\sigma_{\sigma'}T^{\rho\sigma' 1}\equiv 0 \mod I \tag{23}$$

which can be written in matrix form as

$$\overline{U}\,\mathcal{B}+\mathcal{B}\,\overline{U}^{\mathbf{t}}\equiv 0 \mod I, \tag{24}$$

where $\mathcal{B}^{\rho\sigma}=T^{\rho\sigma 1}$.

Equation (23) imposes $k^2-\dim\mathfrak{h}_{\mathcal{B}}$ independent conditions on the space $\langle\overline{U},I\rangle$ when pulled back to $\widetilde{\mathfrak{g}}_T$, where $k=m-1$ and $\mathfrak{h}_{\mathcal{B}}$ is the Lie algebra of the annihilator of $\mathcal{B}$. This is because (23) is an inhomogeneous version of the relations placed on elements of $\mathfrak{gl}(M)^*$ when they are pulled back to $\mathfrak{h}_{\mathcal{B}}$. This gives the upper bound

$$\dim \widetilde{\mathfrak{g}}_T \leq (3(m-1)+2) + \dim \mathfrak{h}_{\mathcal{B}}. \tag{25}$$

In what follows, we use notation from the proof of Lemma 5.1: $m-1 = e+\ell+f$ with $\mathcal{B} = Q+\Lambda$, Q is symmetric of rank $e+\ell$, Λ is skew-symmetric of rank $\ell+f$, and $\mathbb{C}^{m-1} = E \oplus L \oplus F$ with $\ker(Q) = F$ and $\ker(\Lambda) = E$.

Equation (25) implies that if $\dim \mathfrak{h}_{\mathcal{B}} < \binom{m-1}{2} - m$ then T is eliminated from consideration in Theorem A. Thus we are reduced to consider the seven cases described in Lemma 5.1.

By inequality (25) and Lemma 5.1, in order to eliminate a tensor from consideration, it is enough to obtain the additional constraints on $\mathfrak{h}_{\mathcal{B}}$ as follows, where the second line is the number of relations needed to eliminate the case from consideration:

(26)

A1	**A2**	**A3**	**A4**	**B1**	**B2**	**B3**
$2m-1$	m	$m+1$	$m-1$	4	1	2

Rewrite the $(\rho\sigma\tau)$ equation (21) using (15)–(17) to substitute out u_1^ρ, v_1^σ and w_1^τ, and (18)–(19) to substitute out $\overline{U}$ and $\overline{V}$. The result is

$$\begin{aligned} &[-v^1_{\sigma'}T^{\rho\sigma'1} - w^1_\rho]\delta_{\sigma\tau} + [-u^1_{\rho'}T^{\rho'\sigma 1} - w^1_\sigma]\delta_{\rho\tau} + [-u^1_\tau - v^1_\tau]T^{\rho\sigma 1} + \\ &[-v^1_1\delta_{\rho\rho'} - v^1_{\sigma'}T^{\rho\sigma'\rho'} - w^{\rho'}_\rho]T^{\rho'\sigma\tau} + [-u^1_1\delta_{\sigma\sigma'} - u^1_{\rho'}T^{\rho'\sigma\sigma'} - w^{\sigma'}_\sigma]T^{\rho\sigma'\tau} + w^\tau_{\tau'}T^{\rho\sigma\tau'} = 0. \end{aligned} \tag{27}$$

Let $\widehat{T} = T^{\rho\sigma\tau}a_\rho \otimes b_\sigma \otimes c_\tau$ and rewrite (27) as

$$\begin{aligned} &\sum_{\rho'} u^1_{\rho'}[T^{\rho'\sigma 1}\delta_{\rho\tau} + \delta_{\rho'\tau}T^{\rho\sigma 1} + T^{\rho'\sigma\sigma'}T^{\rho\sigma'\tau}] + \sum_{\sigma'} v^1_{\sigma'}[T^{\rho\sigma'1}\delta_{\sigma\tau} + \delta_{\sigma'\tau}T^{\rho\sigma 1} + T^{\rho\sigma'\rho'}T^{\rho'\sigma\tau}] \\ &+ \sum_{\tau'} w^1_{\tau'}[\delta_{\rho\tau'}\delta_{\sigma\tau} + \delta_{\tau'\sigma}\delta_{\rho\tau}] + (u^1_1 + v^1_1)T^{\rho\sigma\tau} = [\overline{W}^{\mathbf{t}}.\widehat{T}]^{\rho\sigma\tau}, \end{aligned} \tag{28}$$

where $[\overline{W}.\widehat{T}]^{\rho\sigma\tau} = w^\rho_{\rho'}\widehat{T}^{\rho'\sigma\tau} + w^\sigma_{\sigma'}\widehat{T}^{\rho\sigma'\tau} - \sum_{\tau'} w^{\tau'}_\tau \widehat{T}^{\rho\sigma\tau'}$ denotes the action of $\mathfrak{gl}_{m-1}$ on $\mathbb{C}^{m-1} \otimes \mathbb{C}^{m-1} \otimes (\mathbb{C}^{m-1})^*$.

Lemma 6.1. *Unless $\widehat{T} = 0$, the tensor T cannot appear in Theorem A.*

Proof. Let $R \subset M = \mathbb{C}^{m-1}$ be the smallest subspace of M such that we may write

$$\widehat{T} = r_1 \otimes \mathrm{Id}_M + \sigma_{12}(r_2 \otimes \mathrm{Id}_M) + \mathcal{B} \otimes \mu_3 + \widehat{T}',$$

where $r_1, r_2 \in R, \mu_3 \in R^*$, and $\widehat{T}' \in R \otimes R \otimes R^*$.

The action of $\mathfrak{h}_{\mathcal{B}}$ on $\widehat{T}$ is the same as the action of $\mathfrak{h}_{\mathcal{B}}$ on $\widetilde{T} := r_1 \oplus r_2 \oplus \mu_3 \oplus \widehat{T}' \in R \oplus R \oplus R^* \oplus (R \otimes R \otimes R^*)$ because Id_M and $\mathcal{B} \in M \otimes M$ are acted on trivially by $\mathfrak{h}_{\mathcal{B}}$.

The Lemma will follow from Lemmas 6.2, 6.3, and 6.4, which will respectively eliminate the cases $\dim R > 2$, $\dim R = 1$, and $\dim R = 2$ from consideration. □

Lemma 6.2. *If $\dim R > 2$, then T cannot appear in Theorem A.*

Proof. Degenerate $\widetilde{T}$ in such a way that it sits inside some $R \oplus R \oplus R^* \oplus (R \otimes R \otimes R^*)$ with $\dim R = 3$, but no smaller dimensional space.

The proof of Lemma 5.1 shows that we have a matrix presentation for elements in $\mathfrak{h}_{\mathcal{B}}$ given by (11) and the further normalizations below it. Observe that in all cases under consideration the minimum number of constraints is given when R is spanned by the first three basis vectors in $\mathbb{C}^{m-1}$. In this case, the annihilator of $\widetilde{T}$ in $\mathfrak{gl}_{m-1}$ is contained in the parabolic $\mathfrak{p}_3 \subset \mathfrak{gl}_{m-1}$, the Lie algebra consisting of matrices with their $(m-4) \times 3$ lower-left block zero.

We will need our hypothesis that $m \geq 14$ in what follows. Intersecting $\mathfrak{gl}_{m-1}$ with $\mathfrak{p}_3$ gives $3(m-4)$ relations on entries of the first three columns in $\mathfrak{gl}_{m-1}$. We now discuss the restrictions these place on $\mathfrak{h}_{\mathcal{B}}$ and compare with (26).

A1. In this case $X = X_{FF}$. The lower-left $(\frac{m-1}{2}) \times (\frac{m-1}{2})$-block of X_{FF} is a symmetric matrix B. Restricting to this submatrix, we see that $B_{1,2} = 0, B_{1,3} = 0, B_{2,3} = 0$ respectively imply $B_{2,1} = 0, B_{3,1} = 0, B_{3,2} = 0$. We obtain $3(m-4) - 3 > 2m - 1$ independent relations on $\mathfrak{h}_{\mathcal{B}}$.

A2. In this case $X = X_{EE}$. The lower-left $(\frac{m-1}{2}) \times (\frac{m-1}{2})$-block of X_{EE} is a skew-symmetric matrix B. Restricting to this submatrix, we see that $B_{1,1} = B_{2,2} = B_{3,3} = 0$ do not give new relations. Moreover, $B_{1,2} = 0$, $B_{1,3} = 0$, $B_{2,3} = 0$ respectively imply $B_{2,1} = 0, B_{3,1} = 0, B_{3,2} = 0$. We obtain $3(m-4) - 6 > m$ independent relations on $\mathfrak{h}_{\mathcal{B}}$.

A3. In this case X_{EE}, X_{LE} do not appear and $X_{LL} = 0$ as $\ell = 1$ and $X_{LL} \in \mathfrak{so}_\ell$. Hence the first column of X is zero. Intersecting $\mathfrak{h}_{\mathcal{B}}$ with $\mathfrak{p}_3$ gives constraints only on the second and third column which yield up to $2(m-4)$ relations. However the lower-left $(\frac{m-2}{2}) \times (\frac{m-2}{2})$-block B of X is a symmetric matrix so $B_{2,1} = B_{3,1} = 0$ imply $B_{1,2} = B_{1,3} = 0$ and $B_{2,3} = 0$ implies $B_{3,2} = 0$, We obtain $2(m-4) - 3 > m+1$ relations.

A4. In this case X_{LL}, X_{LF} do not appear, and $X_{EE} = 0$ as $\ell = 1$ and $X_{EE} \in \mathfrak{so}_\ell$. Hence the first column of X is zero. Thus intersecting $\mathfrak{h}_{\mathcal{B}}$ with $\mathfrak{p}_3$ gives constraints only on the second and third column which yield up to $2(m-4)$ relations. However the lower-left $(\frac{m-2}{2}) \times (\frac{m-2}{2})$-block of X_{FF} is a symmetric matrix B, so $B_{1,2} = 0$ implies $B_{2,1} = 0$. We obtain $2(m-4) - 1 > m - 1$ independent relations.

B Cases. In all **B** cases, intersecting $\mathfrak{h}_{\mathcal{B}}$ with $\mathfrak{p}_3$ gives constraints only on the third column which yield $m - 4 > 4$ relations.

□

We now assume $\dim R \leq 2$.

We fix, case, by case, a non-canonical even-dimensional subspace $V \subset \mathbb{C}^{m-1}$, disjoint from R and with the properties that $\mathcal{B}|_{R \times V} = 0$ and $\mathcal{B}|_V$ is skew and non-degenerate, except for in case **A2** where it is symmetric and non-degenerate. The utility of V is that we may normalize $\mathcal{B}|_V$. We may choose V such that $\dim V$ is respectively at least $m - 1 - 2\dim(R)$, $m - 2 - 2\dim(R)$, $m - 3 - 2\dim(R)$, $m - 2 - 2\dim(R)$, in cases **A1-A4** and of slightly smaller dimensions in the **B** cases.

In the special cases $\dim R = 1$ and $R = L$ (resp. $R = E$) in case **A3** (resp. **A4**) we have $\dim V \geq m - 4$ (resp. $\dim V \geq m - 2$).

Let ξ, ϕ, ψ, π run over the first half of the indices of V and their overlines the second half. We may then write $\mathcal{B}|_V = \sum a_\xi b_{\overline{\xi}} + \epsilon a_{\overline{\xi}} b_\xi$, where $\epsilon = -1$ except in case **A2** where $\epsilon = 1$.

Lemma 6.3. *If* $\dim R = 1$*, then* T *cannot appear in Theorem A.*

Proof. Intersecting the parabolic $\mathfrak{p}_1$ (which by definition stabilizes R) and the Lie algebra $\mathfrak{h}_{\mathcal{B}}$, determines additional conditions on $\mathfrak{h}_{\mathcal{B}}$ in cases **A1** and **A2**, where we have $m-2$ and $m-3$ extra conditions respectively. Thus the number of further relations on $\langle \mathfrak{h}_{\mathcal{B}}, I \rangle$ we are required to find in the seven cases to eliminate them are respectively

A1	**A2**	**A3**	**A4**	**B1**	**B2**	**B3**
$m+1$	3	$m+1$	$m-1$	4	1	2

Here, in cases **A3**, **A4** we assume R has the worst position possible, namely in case **A3** $R = L$ and in case **A4** that $R = E$. Otherwise we only need 5 (resp. 4) further relations to exclude these cases, which we label **A3'**, **A4'**.

Use the index x for the space R, $\overline{x}$ for the space R^c such that $\Lambda|_{R \times R^c}$ is nondegenerate (except in case **A2** where it is the space such that $Q|_{R\times R^c}$ is nondegenerate. In case **A3**, use $\widehat{x}$ for the space such that $Q|_{R\times R^c}$ is nondegenerate.

We have $T^{\rho\sigma\tau} = 0$, except possibly: $T^{x\sigma\sigma}$ (this is independent of σ, for $\sigma \neq x$), $T^{\rho x \rho}$ (this is independent of ρ, for $\rho \neq x$), T^{xxx}, and for cases other than **A3**, $T^{\rho\overline{\rho}x}$, where $\overline{\rho}$ is the unique index such that $T^{\rho\overline{\rho}1} \neq 0$. In case **A3**, for the unique index in L, there two such indices. Our normalizations imply $T^{\phi\overline{\phi}1} = 1, T^{\overline{\phi}\phi 1} = \epsilon$.

Let $z_1 = T^{x\psi\psi} = T^{x\overline{\psi\psi}}$, $z_2 = T^{\psi x \psi} = T^{\overline{\psi} x \overline{\psi}}$, $z_3 = T^{\psi\overline{\psi}x} = \epsilon T^{\overline{\psi}\psi x}$, and $z = T^{xxx}$.

Consider the equations

$$(\phi\psi\psi) \quad -v^1_\phi(1 + z_3 z_1) - w^1_\phi - w^x_\phi z_1 = 0, \tag{29}$$

$$(\psi\phi\psi) \quad -\epsilon u^1_\phi(1 + z_3 z_2) - w^1_\phi - w^x_\phi z_2 = 0, \tag{30}$$

$$(\overline{\phi\psi\psi}) \quad -\epsilon v^1_\phi(1 + z_3 z_1) - w^1_{\overline{\phi}} - w^x_{\overline{\phi}} z_1 = 0, \tag{31}$$

$$(\overline{\psi\phi\psi}) \quad -u^1_\phi(1 + z_2 z_3) - w^1_{\overline{\phi}} - w^x_{\overline{\phi}} z_2 = 0. \tag{32}$$

If $z_1 = z_2 = 0$, this collection of four equations gives at least $2\dim V$ independent equations. By our lower bound on $\dim V$, we conclude that if $z_1 = z_2 = 0$, all cases are eliminated.

Henceforth we assume $(z_1, z_2) \neq (0,0)$. Consider $(29)\cdot z_2 - (30)\cdot z_1$ and $(31)\cdot z_2 - (32)\cdot z_1$, which respectively yield:

$$-v^1_{\overline{\phi}}(1 + z_3 z_1) z_2 + \epsilon u^1_{\overline{\phi}}(1 + z_3 z_2) z_1 + w^1_\phi (z_1 - z_2) = 0, \tag{33}$$

$$-\epsilon v^1_\phi(1 + z_3 z_1) z_2 + u^1_\phi(1 + z_2 z_3) z_1 + w^1_{\overline{\phi}}(z_1 - z_2) = 0. \tag{34}$$

Plugging in (18) and (19) into (20) transforms the $(\rho\tau 1)$ equation to:

$$-(w^\tau_\rho + v^1_1\delta_{\rho\tau} + v^1_{\sigma'}T^{\rho\sigma'\tau})T^{\tau\sigma 1} - (w^\tau_\sigma + u^1_1\delta_{\sigma\tau} + u^1_{\rho'}T^{\rho'\sigma\tau})T^{\rho\tau 1} + w^1_1 T^{\rho\sigma 1} + w^1_\tau T^{\rho\sigma\tau} = 0. \tag{35}$$

Using (35), we obtain:

$$(\overline{\phi}\overline{x}1) \quad -v^1_\phi z_3\epsilon - w^x_{\overline{\phi}} - w^\phi_{\overline{x}}\epsilon = 0, \tag{36}$$

$$(\overline{x}\overline{\phi}1) \quad -\epsilon u^1_\phi z_3 - \epsilon w^x_{\overline{\phi}} - w^\phi_{\overline{x}} = 0, \tag{37}$$

$$(\phi\overline{x}1) \quad -v^1_{\overline{\phi}} z_3 - w^x_\phi - w^{\overline{\phi}}_{\overline{x}} = 0, \tag{38}$$

$$(\overline{x}\phi1) \quad -u^1_{\overline{\phi}} z_3 - \epsilon w^x_\phi - w^{\overline{\phi}}_{\overline{x}}\epsilon = 0. \tag{39}$$

These imply:

$$(-\epsilon u^1_{\overline{\phi}} + v^1_{\overline{\phi}})z_3 = 0, \tag{40}$$

$$(u^1_\phi - \epsilon v^1_\phi)z_3 = 0. \tag{41}$$

Now, if $z_3 \neq 0$, (resp. $z_3 = 0$), then (40),(41) (resp. (33),(34)) provide $\dim V$ relations. We conclude that cases **A2**, **A3'**, **A4'** and the **B** cases are eliminated when $\dim R = 1$.

If $z_1 \neq z_2$ and $z_3 \neq 0$, then (40) and (41) are linearly independent from (33) and (34). Thus we obtain at least $2\dim V$ relations eliminating this scenario in all cases.

Consider the following equations from (35), where T^{xx1} is zero in case **A1** and it may or may not be zero in case **A3**:

$$(\phi x1) \quad -v^1_{\overline{\phi}} z_3 T^{xx1} - u^1_{\overline{\phi}} z_2 + w^1_\phi z_2 - w^{\overline{\phi}}_x - w^{\overline{x}}_\phi \epsilon = 0, \tag{42}$$

$$(x\phi1) \quad -v^1_{\overline{\phi}} z_1\epsilon + w^1_\phi z_1 - u^1_{\overline{\phi}}\epsilon z_3 T^{xx1} - w^{\overline{\phi}}_x\epsilon - w^{\overline{x}}_\phi = 0, \tag{43}$$

$$(\overline{\phi}x1) \quad -v^1_\phi \epsilon z_3 T^{xx1} - u^1_\phi z_2\epsilon + w^1_{\overline{\phi}} z_2 - w^{\overline{x}}_{\overline{\phi}}\epsilon - w^\phi_x\epsilon = 0, \tag{44}$$

$$(x\overline{\phi}1) \quad -v^1_\phi z_1 - u^1_\phi z_3 T^{xx1} + w^1_{\overline{\phi}} z_1 - w^\phi_x - w^{\overline{x}}_{\overline{\phi}} = 0. \tag{45}$$

Combining (42) with (43), and (44) with (45) (using $\epsilon = -1$) gives

$$-v^1_{\overline{\phi}} z_3 T^{xx1} + u^1_{\overline{\phi}} z_3 T^{xx1} - u^1_{\overline{\phi}} z_2 - v^1_{\overline{\phi}} z_1 - w^1_\phi(z_1+z_2) = 0, \tag{46}$$

$$+v^1_\phi z_3 T^{xx1} - u^1_\phi z_3 T^{xx1} + u^1_\phi z_2 + v^1_\phi z_1 - w^1_{\overline{\phi}}(z_1+z_2) = 0. \tag{47}$$

We see that if $z_3 \neq 0$, either (46),(47), (40),(41) or (33),(34), (40),(41) provide enough equations to eliminate all cases, namely $2\dim(V)$ equations, i.e., $2(m-3)$ in case **A1** and $2(m-7)$ in cases **A3**,**A4**.

So from now on assume $z_3 = 0$, so (46),(47) become

$$-u^1_{\overline{\phi}} z_2 - v^1_{\overline{\phi}} z_1 - w^1_\phi(z_1+z_2) = 0, \tag{48}$$

$$+u^1_\phi z_2 + v^1_\phi z_1 - w^1_{\overline{\phi}}(z_1+z_2) = 0. \tag{49}$$

We see (46),(47)(33),(34) provide $2\dim(V)$ equations unless $z_1^2 + z_2^2 = 0$, in which case they provide $\dim(V)$ equations.

So from now on, assume $z_1^2 + z_2^2 = 0$. The table of cases that remain and number of equations needed to eliminate is now as follows.

A1	A3	A4
4	3	1

We consider the following equations (recall $z = T^{xxx}$):

$$(\phi x x) \quad -v^1_{\overline{\phi}} - w^1_{\phi} + w^x_{\phi}(z_2 - z) = 0, \tag{50}$$

$$(x\phi x) \quad u^1_{\overline{\phi}} - w^1_{\phi} + w^x_{\phi}(z_1 - z) = 0. \tag{51}$$

From $(\phi xx)\cdot(z_1 - z) - (x\phi x)\cdot(z_2 - z)$, we derive:

$$u^1_{\overline{\phi}}(z - z_2) + v^1_{\overline{\phi}}(z - z_1) + w^1_{\phi}(z_2 - z_1) = 0. \tag{52}$$

Similarly, from $(\overline{\phi}xx)$ and $(x\overline{\phi}x)$, we obtain:

$$u^1_{\phi}(z - z_2) + v^1_{\phi}(z - z_1) + w^1_{\overline{\phi}}(z_1 - z_2) = 0. \tag{53}$$

We consider the 2×3 coefficient matrix corresponding to the equations (33) and (52), with respect to the variables $u^1_{\overline{\phi}}, v^1_{\overline{\phi}}, w^1_{\phi}$, and the 2×3 coefficient matrix corresponding to (34) and (53), with respect to $u^1_{\phi}, v^1_{\phi}, w^1_{\overline{\phi}}$. All the 2×2 minors of both of these matrices are up to sign equal to $-z_1^2 + z_2^2 + z(z_1 - z_2)$. Thus we are done unless $z = \frac{2z_1^2}{(z_1 - z_2)}$.

We are reduced to the case $z_3 = 0$, $z_2 = \widehat{i} z_1$ where $\widehat{i} = \pm i$, and $z = \frac{2}{(1-\widehat{i})} z_1$. Consider, in the **A1** case:

$$\begin{aligned}
(xx1) &\quad v^1_{\overline{x}} - \widehat{i} u^1_{\overline{x}} + \frac{2}{1-\widehat{i}} w^1_{x} = 0,\\
(\overline{x}x1) + (x\overline{x}1) &\quad \frac{2}{1-\widehat{i}} u^1_{x} + \frac{2}{1-\widehat{i}} v^1_{x} + u^1_{\overline{x}} - v^1_{\overline{x}} + (1+\widehat{i}) w^1_{\overline{x}} = 0,\\
\widehat{i}(\overline{xxx}) + (\overline{xxx}) &\quad \frac{1+\widehat{i}}{1-\widehat{i}} z_1^2 u^1_{x} + \frac{-1+\widehat{i}}{1-\widehat{i}} z_1^2 v^1_{x} + (2\widehat{i} - 1) u^1_{\overline{x}} + (\widehat{i} - 1) v^1_{\overline{x}} - (1+\widehat{i}) w^1_{x} = 0,\\
(3+\widehat{i})((x\overline{x}x) + (1+\widehat{i})(\overline{x}xx) &\quad -(5+\widehat{i}) u^1_{x} - (1-\widehat{i}) v^1_{x} - (4+2\widehat{i}) w^1_{\overline{x}} = 0.
\end{aligned}$$

The matrix of coefficients for these four equations (in the variables $u^1_x, v^1_x, u^1_{\overline{x}}, v^1_{\overline{x}}, w^1_x, w^1_{\overline{x}}$) is as follows:

$$\begin{pmatrix}
0 & 0 & -\widehat{i} & 1 & \frac{2}{1-\widehat{i}} & 0\\
-\frac{2}{1-\widehat{i}} & \frac{2}{1-\widehat{i}}1 & 1 & -1 & 0 & (1+\widehat{i})\\
\frac{1+\widehat{i}}{1-\widehat{i}} z_1^2 & \frac{-1+\widehat{i}}{1-\widehat{i}} z_1^2 & (2\widehat{i}-1) & (\widehat{i}-1) & -(1+\widehat{i}) & 0\\
-(5+\widehat{i}) & (-1+\widehat{i}) & 0 & 0 & 0 & -(4+2\widehat{i})
\end{pmatrix}.$$

The last 4×4 minor does not involve z_1 and it is nonzero, so we conclude in the **A1** case. The other cases are similar except in the **A3** case, one of $T^{\widehat{x}x1}$ T^{xx1} will appear (the first in the case $R^c \neq R$, the second in the case $R^c = R$), and in the **A4** case T^{xx1} appears. □

Lemma 6.4. *If* $\dim R = 2$*, then* T *cannot appear in Theorem A.*

Proof. Let x, y denote the two indices of R. The intersection of the parabolic $\mathfrak{p}_2$ (which stabilizes R) and the Lie algebra $\mathfrak{h}_{\mathcal{B}}$ implies the following conditions on $\mathfrak{h}_{\mathcal{B}}$: in **A** cases we have $2m-3, 2m-9, m-3, m-3$ further restrictions respectively, and none for the **B** cases. Thus the number of independent relations on $\langle \mathfrak{h}_{\mathcal{B}}, I\rangle$ we are required to find are

A1	**A2**	**A3**	**A4**	**B1**	**B2**	**B3**
2	0	4	2	4	1	2

All cases are resolved whenever we find at least four relations. Introduce the notation $z_1 = T^{x\psi\psi}$, $z_1' = T^{y\psi\psi}$, $z_2 = T^{\psi x\psi}$, $z_2' = T^{\psi y\psi}$;

We use the equations:

$$(\phi\psi\psi) \quad -v^1_{\bar\phi}(1 + T^{\phi\bar\phi x}T^{x\psi\psi} + T^{\phi\bar\phi y}T^{y\psi\psi}) - w^1_{\bar\phi} - w^x_{\phi}T^{x\psi\psi} - w^y_{\phi}T^{y\psi\psi} = 0, \tag{54}$$

$$(\psi\phi\psi) \quad -u^1_{\bar\phi}(\epsilon + T^{\bar\phi\phi x}T^{\psi x\psi} + T^{\phi\bar\phi y}T^{y\psi\psi}) - w^1_{\bar\phi} - w^x_{\phi}T^{\psi x\psi} - w^y_{\phi}T^{\psi y\psi} = 0. \tag{55}$$

Consider the matrix $Z = \begin{pmatrix} z_1 & z_1' \\ z_2 & z_2' \end{pmatrix}$. If Z is not invertible, then some linear combination of these equations gives a relation among $v^1_{\bar\phi}, w^1_{\bar\phi}, u^1_{\bar\phi}$ for each ϕ and we conclude. Otherwise, use the system to express for w^x_{ϕ} and w^y_{ϕ} in terms of elements in I. Then use the conjugate equations to express $w^x_{\bar\phi}$ and $w^y_{\bar\phi}$ in terms of elements in I. Finally substitute these into the $(\bar\phi\bar x 1)$, and $(\bar x\bar\phi 1)$, eliminate the $w^{\phi}_{\bar x}$ term to get relations among the elements of I to conclude. $\square$

Conclusion of the case $T(\gamma^1)$ has full rank. To conclude, it is now straight-forward to compute $\dim \mathfrak{g}_T$ in each of the four possible cases. The only one with a large enough dimension is case **A1**, where $\dim \mathfrak{g}_T = \dim G_T = \binom{m+1}{2} - 1$, as desired, the case $T_{max-1,odd,m}$. Explicitly, for $T_{max-1,odd,m}$ we have:

$$\widetilde{\mathfrak{g}}_{T_{max-1,odd,m}} =$$
$$\left\{ \lambda \mathrm{Id} + \begin{pmatrix} 0 & -\mathbf{y} \ \ \mathbf{x} \\ 0 & -Z^{\mathbf{t}} \end{pmatrix}, \mu \mathrm{Id} + \begin{pmatrix} 0 & \mathbf{y} \ \ -\mathbf{x} \\ 0 & -Z^{\mathbf{t}} \end{pmatrix}, \nu \mathrm{Id} + \begin{pmatrix} 0 & 0 \\ \mathbf{x}^{\mathbf{t}} & \\ \mathbf{y}^{\mathbf{t}} & Z \end{pmatrix} \mid \begin{array}{c} \lambda+\mu+\nu = 0, \\ \mathbf{x}, \mathbf{y} \in \mathbb{C}^q, \\ Z \in \mathfrak{sp}(m-1) \end{array} \right\}.$$

7. Case $T(\gamma^1)$ drops rank

Start with the normalizations as in the binding case. An explicit calculation shows that the freedom to normalize further after the normalization $T^{1jk} = \delta_{jk}$ and $T^{i1k} = \delta_{ik}$ is the subgroup of $GL(A) \times GL(B) \times GL(C)$ defined by

$$\left\{ \left(\begin{pmatrix} x^1_1 & -\mathbf{z}^{\mathbf{t}}(\overline{Z}^{\mathbf{t}})^{-1} \\ 0 & x^1_1(\overline{Z}^{\mathbf{t}})^{-1} \end{pmatrix}, \begin{pmatrix} y^1_1 & -\mathbf{z}^{\mathbf{t}}(\overline{Z}^{\mathbf{t}})^{-1} \\ 0 & y^1_1(\overline{Z}^{\mathbf{t}})^{-1} \end{pmatrix}, \begin{pmatrix} z^1_1 & 0 \\ \mathbf{z} & z^1_1\overline{Z} \end{pmatrix} \mid x^1_1 y^1_1 z^1_1 = 1, \mathbf{z} \in \mathbb{C}^{m-1}, \overline{Z} \in GL_{m-1} \right) \right\}. \tag{56}$$

Since we can add arbitrary multiples of γ^j's to γ^m, we may assume $T(\gamma^m)$ is full rank (which means we may no longer have normalizations on the third factor by g with $g^m_j \neq 0$, $1 \leq j \leq m-1$). Introduce the additional index range $2 \leq s, t, u \leq m-1$. We have $T(\gamma^m) = a_1 \otimes b_m + a_m \otimes b_1 + T^{\rho\sigma m} a_\rho \otimes b_\sigma$, with $T^{stm} a_s \otimes b_t$ of maximal rank $m-2$.

We now have $A \simeq B \simeq C^* = L_1 \oplus N \oplus L_m$, so there are 27 components of $A \otimes B \otimes C$ to examine instead of just 8 in the previous case. Thirteen of them are simple thanks to our normalizations, which enable us to define an initial ideal I of dimension $5m-4$. We then use the bilinear form on $N \otimes N \otimes L_m^*$ to get $\mathfrak{h}_{\mathcal{B}} \in N \otimes N^*$, and the bound $\dim \mathfrak{g}_T \leq \dim \mathfrak{h}_{\mathcal{B}} + 5m - 4$. As before we then begin to analyze the remaining components of T. The computation for the $N \otimes N \otimes N^*$ is similar to (and easier than) that of the case $T(\gamma^1)$ full rank and is omitted. There remains 12 additional components to analyze, and we carry out the analysis to conclude.

The first 13 components of $(L_1 \oplus N \oplus L_m)^{\otimes 2} \otimes (L_1 \oplus N \oplus L_m)^*$ are:

$$\begin{aligned}
&(111) && u_1^1 + v_1^1 + w_1^1 = 0,\\
&(11m) && u_m^1 + v_m^1 + w_1^m = 0,\\
&(1mm) && u_1^1 + u_\rho^1 T^{\rho mm} + v_m^m + w_m^m = 0,\\
&(m1m) && u_m^m + v_1^1 + v_\sigma^1 T^{m\sigma m} + w_m^m = 0,\\
&(11t) && u_t^1 + v_t^1 + w_1^t = 0,\\
&(1sm) && u_\rho^1 T^{\rho sm} + v_m^s + w_s^m = 0,\\
&(s1m) && u_m^s + v_\sigma^1 T^{s\sigma m} + w_s^m = 0,\\
&(1mt) && u_\rho^1 T^{\rho mt} + v_t^m + w_m^t = 0,\\
&(m1t) && u_t^m + v_\sigma^1 T^{m\sigma t} + w_m^t = 0,\\
&(1s1) && u_\rho^1 T^{\rho s1} + v_1^s + w_s^1 = 0,\\
&(t11) && u_1^t + v_\sigma^1 T^{t\sigma 1} + w_t^1 = 0,\\
&(1m1) && u_\rho^1 T^{\rho m1} + v_1^m + w_m^1 = 0,\\
&(m11) && u_1^m + v_\sigma^1 T^{m\sigma 1} + w_m^1 = 0.
\end{aligned} \tag{57}$$

These imply that the following 13 types of elements

$$w_1^1, u_m^m, v_m^m, w_1^s, v_1^s, u_1^s, u_m^s, v_m^s, u_t^m, v_t^m, u_1^m, v_1^m, w_1^m$$

are determined by $I := \langle u_1^1, v_1^1, w_m^m, u_\rho^1, v_\rho^1, w_\rho^1, w_s^m, w_m^t \rangle$, which has dimension $5m-4$.

The equations

$$\begin{aligned}
&(1st) && u_1^1 \delta_{st} + u_\rho^1 T^{\rho st} + v_t^s + w_s^t = 0,\\
&(s1t) && u_t^s + v_1^1 \delta_{st} + v_\sigma^1 T^{s\sigma t} + w_s^t = 0,
\end{aligned} \tag{58}$$

substituted into the (stm) term

$$(stm) \quad u_\rho^s T^{\rho tm} + v_\sigma^t T^{s\sigma m} + w_m^m T^{stm} + w_1^m T^{st1} + w_u^m T^{stu} = 0$$

give

$$u_{s'}^s T^{s'tm} + u_m^s T^{mtm} + v_{t'}^t T^{st'm} + v_m^t T^{smm} + w_m^m T^{stm} + w_1^m T^{st1} + w_u^m T^{stu} = 0.$$

Using (57) we have

$$\begin{aligned}
&-[w_s^{s'} + v_1^1 \delta_{ss'} + v_\sigma^1 T^{s\sigma s'}]T^{s'tm} - [v_\sigma^1 T^{s\sigma m} + w_s^m]T^{mtm} - [w_t^{t'} + u_1^1 \delta_{tt'} + u_\rho^1 T^{\rho tt'}]T^{st'm}\\
&-[u_\rho^1 T^{\rho tm} + w_t^m]T^{smm} + w_m^m T^{stm} - [u_m^1 + v_m^1]T^{st1} + w_u^m T^{stu} = 0.
\end{aligned}$$

Collecting the w^s_t terms, writing $\widehat{W} = w^s_t a_s \otimes \alpha^t$ and $\mathcal{B} = T^{stm} a_s \otimes a_t$, this may be written

$$[\widehat{W}\mathcal{B} + \mathcal{B}\widehat{W}^{\mathfrak{t}}]^{stm} = v^1_\sigma T^{s\sigma s'} T^{s'tm} - [v^1_\sigma T^{s\sigma m} + w^m_s] T^{mtm} + u^1_\rho T^{\rho t t'} T^{st'm} + (u^1_1 + v^1_1) T^{stm} - [u^1_\rho T^{\rho t m} + w^m_t] T^{smm} + w^m_m T^{stm} - [u^1_m + v^1_m] T^{st1} + w^m_u T^{stu}$$

which implies

$$[\widehat{W}\mathcal{B} + \mathcal{B}\widehat{W}^{\mathfrak{t}}]^{stm} \equiv 0 \operatorname{mod} I' := \langle u^1_j, v^1_j, w^m_\rho, w^1_m \rangle$$

Thus, modulo I', $\widehat{W}$ is $\mathfrak{h}_{\mathcal{B}}$-valued where $\mathfrak{h}_{\mathcal{B}}$ is the Lie algebra annihilating the bilinear form $\mathcal{B}$. Since $\dim I = 5m - 4$, the calculations we have already done show

$$\dim \mathfrak{g}_T \leq \dim \mathfrak{h}_{\mathcal{B}} + 5m - 4 - 2. \tag{59}$$

At this point all **B** cases are eliminated and the number of further relations on $\langle \mathfrak{h}_{\mathcal{B}}, I\rangle$ we are required to find in the four cases to eliminate them are respectively

A1	**A2**	**A3**	**A4**
$3m-4$	$2m-1$	$2m$	$2m-1$

Using (63) in the cases $(stu) = (s\bar{s}u)$ with $u \neq s, \bar{s}$ and $(stu) = (stt)$ with $t \neq s, \bar{s}$, we obtain

(60)

$$(s\bar{s}u) \quad w^u_m = (u^1_u + v^1_u) T^{s\bar{s}1} + (v^1_{\bar{s}} T^{s\bar{s}m} + v^1_m T^{smm} + w^m_s) T^{m\bar{s}u} + (u^1_s T^{s\bar{s}m} + u^1_m T^{msm} + w^m_s) T^{smu} - (v^1_1 \delta_{ss'} + v^1_\sigma T^{s\sigma s'} + w^{s'}_s) T^{s'\bar{s}u} - (u^1_1 \delta_{\bar{s}t} + u^1_\rho T^{\rho\bar{s}t} + w^t_{\bar{s}}) T^{stu} + w^{u'}_u T^{s\bar{s}u'}$$

(61)

$$(stt) \quad w^1_s = -v^1_\rho T^{s\rho 1} - (v^1_{\bar{s}} T^{s\bar{s}m} + v^1_m T^{smm} + w^m_s) T^{mtt} - (u^1_{\bar{t}} T^{\bar{t}tm} + u^1_m T^{mtm} + w^m_t) T^{smt} - (u^1_t + v^1_t) T^{st1} - (v^1_1 \delta_{ss'} + v^1_\sigma T^{s\sigma s'} + w^{s'}_s) T^{s'tt} - (u^1_1 \delta_{\bar{s}t'} + u^1_\rho T^{\rho t t'} + w^{t'}_t) T^{st't} + w^{u'}_t T^{stu'}.$$

(Note that (tst) gives a similar equation to (stt) with the roles of u and v exchanged.) Recalling that $w^s_t \equiv 0 \operatorname{mod} \langle I', \mathfrak{h}_{\mathcal{B}}\rangle$, we have $2(m-2)$ new relations among elements of $I, \mathfrak{h}_{\mathcal{B}}$. In other words, $\widetilde{\mathfrak{g}}_T$ is spanned by $\langle \mathfrak{h}_{\mathcal{B}}, I'\rangle$. The number of further relations on $\langle \mathfrak{h}_{\mathcal{B}}, I'\rangle$ we are required to find in the four cases to eliminate them are respectively

A1	**A2**	**A3**	**A4**
m	3	4	3.

Let $\check{T} = T^{stu} a_s \otimes b_t \otimes c_u$ and define $\widehat{W}.\check{T}$ via the action of $\mathfrak{gl}_{m-2}$ on $(\mathbb{C}^{m-2})^{\otimes 3}$ as before. Similar to the case $T(\gamma^1)$ is full rank, we obtain

$$(stu) \quad [\widehat{W}.\check{T}]^{stu} \equiv 0 \operatorname{mod} I'. \tag{62}$$

We omit the proof that $\check{T} = 0$, as it is similar to the proof in the case $T(\gamma^1)$ is of full rank, and the proof is easier because we have far fewer relations that it needs to impose to conclude.

Using $\check{T} = 0$, the (stu) equation becomes

$$(stu) \quad -(w^1_s + v^1_\rho T^{s\rho 1}) \delta_{tu} - (w^1_t + u^1_\rho T^{\rho t 1}) \delta_{su} + w^u_m T^{stm} - [v^1_\sigma T^{s\sigma m} + w^m_s] T^{mtu} - [u^1_\rho T^{\rho t m} + w^m_t] T^{smu} - (u^1_u + v^1_u) T^{st1} = 0. \tag{63}$$

Notational warning: in what follows, in the **A3** case, the equations below are missing a term when one of the indices is equal to 2, $\overline{2}$, or $\widehat{2}$ (in the notation of the case $\dim R = 1$ in Lemma

6.3). Also note that in the **A4** case, and the **A2** case when $m-2$ is odd, $\overline{2}=2$. So in what follows, just avoid the $s=2$ index in those cases. (We will have an abundance of conditions so that it won't be needed.)

We now make the dependencies obtained above in the cases $(stu)=(s\overline{s}u)$ with $u\neq s,\overline{s}$ and $(stu)=(stt)$ with $t\neq s,\overline{s}$, explicit:

(64)

$$(s\overline{s}u)\quad w^u_m=(u^1_u+v^1_u)T^{s\overline{s}1}+(v^1_{\overline{s}}T^{s\overline{s}m}+v^1_mT^{smm}+w^m_s)T^{m\overline{s}u}+(u^1_sT^{s\overline{s}m}+u^1_mT^{msm}+w^m_s)T^{smu}$$

(65)

$$(stt)\quad w^1_s=-v^1_\rho T^{s\rho 1}-(v^1_{\overline{s}}T^{s\overline{s}m}+v^1_mT^{smm}+w^m_s)T^{mtt}-(u^1_{\overline{t}}T^{\overline{t}tm}+u^1_mT^{mtm}+w^m_t)T^{smt}$$

(66)

$$-(u^1_t+v^1_t)T^{st1}.$$

In case **A1**, write

$$\begin{aligned} z^\phi_\psi&=\tfrac12(w^\phi_\psi+w^{\overline{\psi}}_{\overline{\phi}}) & h^\phi_\psi&=\tfrac12(w^\phi_\psi-w^{\overline{\psi}}_{\overline{\phi}}),\\ z^{\overline{\phi}}_\psi&=\tfrac12(w^{\overline{\phi}}_\psi-w^{\overline{\psi}}_\phi) & h^{\overline{\phi}}_\psi&=\tfrac12(w^{\overline{\phi}}_\psi+w^{\overline{\psi}}_\phi),\\ z^{\psi}_{\overline{\phi}}&=\tfrac12(w^{\psi}_{\overline{\phi}}-w^{\phi}_{\overline{\psi}}) & h^{\psi}_{\overline{\phi}}&=\tfrac12(w^{\psi}_{\overline{\phi}}+w^{\phi}_{\overline{\psi}}). \end{aligned}$$

Then

$$\mathcal{I}:=\langle u^1_1,v^1_1,w^m_m,u^1_\rho,v^1_\rho,w^1_m,w^m_s,h^\phi_\psi,h^{\overline{\phi}}_\psi,h^\psi_{\overline{\phi}}\rangle.$$

For the other cases one similarly writes

$$\mathcal{I}:=\langle u^1_1,v^1_1,w^m_m,u^1_\rho,v^1_\rho,w^1_m,w^m_s,\mathfrak{h}_{\mathcal{B}}\rangle.$$

In each case, $\mathfrak{h}_{\mathcal{B}}$ can be explicitly written as $\dim\mathfrak{h}_{\mathcal{B}}$ linear combinations of the w^s_t.

Assume $\mathcal{B}$ has been normalized as in the proof of Lemma 5.1 and consider

$$(67)\qquad (st1)\quad u^s_uT^{ut1}+u^s_mT^{mt1}+v^t_uT^{su1}+v^t_mT^{sm1}+w^1_1T^{st1}+w^1_mT^{stm}=0.$$

This is an equation of the form $\widehat{U}\mathcal{B}^1+\mathcal{B}^1\widehat{U}^{\mathbf{t}}\equiv 0 \bmod I$, where $\mathcal{B}^1:=T^{st1}a_s\otimes a_t$. If we are in case **A1** and $\mathcal{B}^1$ is nonzero, if it has a nonzero symmetric part, it causes a drop of dimension at least $m-4$ (because the smallest orbit is the Veronese), if it has a nonzero skew part which is not a multiple of Λ, it causes a drop of dimension at least $2(m-2)-5$ (the dimension of the isotropic Grassmannian of 2-planes), so we conclude that we may normalize $\mathcal{B}^1$ to $\mathcal{B}^1=T^{221}a_2\otimes b_2+\lambda\Lambda$. In case **A2** we conclude $\mathcal{B}^1=\lambda Q$ because if it has a nonzero symmetric part that is not a multiple of Q, the dimension must drop by at least $m-4$, and if it has a nonzero skew-symmetric part, the dimension must drop by at least $2((m-2)-4)+3=2m-9$. For cases **A3**, **A4**, we are similarly reduced to $\mathcal{B}^1=T^{221}a_2\otimes b_2+\lambda\Lambda$. Here we are just saying that $\mathcal{B}^1$ just comes from the bilinear form, but in cases **A3,A4** the bilinear form has two components. All these conclusions were by analyzing additional restrictions on $\mathfrak{h}_{\mathcal{B}}$. Below we will show that there are further restrictions on I' in all cases that imply $\mathcal{B}^1=0$.

Reconsider (stu) with $s,t\neq u$ and $t\neq\overline{s}$

$$(68)\quad -[v^1_{\overline{s}}T^{s\overline{s}m}+v^1_mT^{smm}+w^m_s]T^{mtu}-[u^1_{\overline{t}}T^{\overline{t}tm}+u^1_mT^{mtm}+w^m_t]T^{smu}-(u^1_u+v^1_u)T^{st1}=0$$

We now use our normalization of T^{st1}.

Setting $s=t=2$, in all but the **A1** case, we see T^{221} must be zero because (68) gives at least $m-4$ equations on the variables $u^1_u+v^1_u$. In the **A1** case we see if $T^{221}\neq 0$, the (222) equation gives an additional restriction and thus if it is nonzero and there are 2 further relations it is eliminated.

In what follows we present the proof that all coefficients $T^{\rho\sigma 1}$ are zero in cases **A1,A3,A4**, and leave the **A2** case to the reader.

Let ϕ, ψ run the first half of the index range of Λ and $\overline{\phi}, \overline{\psi}$ the second half. The equations $T^{s\bar{s}1} = -T^{\bar{s}s1}$ give $m-2$ (resp. $m-3$) additional equations on $\mathcal{I}$ in the **A1** (resp. **A3**, **A4**) cases. In all cases but **A1**, we also obtain $\lambda = 0$ by comparing the $(\phi\overline{\phi}u)$ equation to the $(\overline{\phi}\phi u)$ equation Moreover, in the **A1** case, we may use $(\phi\overline{\psi\psi})$ and $(\overline{\psi}\phi\overline{\psi})$ to obtain $\frac{m}{2}-1$ additional equations for a total of $\frac{3m}{2}-3$ equations if $\lambda \neq 0$ to see $\lambda = 0$ in the **A1** case as well.

Similarly if some $T^{mtu} \neq 0$ for $t \neq u$ we again obtain $m-3$ more relations among the elements of $\mathcal{I}$, and similarly for T^{smu}. If some T^{mtt} or T^{tmt} is nonzero, we obtain $m-2$ more relations among the elements of $\mathcal{I}$ (in addition to the (stt) equation, use the (sss) equation).

In summary, at this point:

In cases **A3**,**A4**, $T^{st1} = 0$, $T^{mst} = 0$ and $T^{smt} = 0$ for all s, t.

In case **A1**, there is at most one among T^{221}, T^{mtu} for some fixed t, u, or T^{smu} for some fixed s, u that is nonzero. If any of these occur, and we find 2 (cases T^{221},T^{mtt},T^{tmt}) or 3 (other cases) additional relations among the elements of $\mathcal{I}$, then T is eliminated from consideration.

Lemma 7.1. *In case* **A1**, $T^{st1} = 0$, $T^{mst} = 0$ *and* $T^{smt} = 0$ *for all* s, t.

Proof. The (stm) equation is

$$z_s^{\bar{t}} = (v_1^1 \delta_{s\bar{t}} + v_m^1 T^{smt}) T^{\bar{t}tm} - u_m^s T^{mtm} + (u_1^1 \delta_{s\bar{t}} + u_m^1 T^{mst}) T^{s\bar{s}m} - v_m^t T^{smm} + w_m^m T^{stm} - w_1^m T^{st1}.$$

In particular, it expresses $z_s^{\bar{t}}$ in terms of elements of $\mathcal{I}$.

If $T^{221} \neq 0$, the $(\phi 21)$ equation with $\phi \neq 2$ becomes

$$-(h_\phi^2 + z_\phi^2 + v_1^1 \delta_{s2}) T^{221} - (v_{\bar{s}}^1 T^{s\bar{s}m} + w_s^m) T^{mt1} - (u_{\bar{2}}^1 T^{\bar{2}2m} + w_2^m) T^{sm1} + w_m^1 T^{s2m} = 0. \tag{69}$$

We obtain $\frac{m}{2}-1$ additional equations expressing the h_ϕ^2 in terms of elements of $\mathcal{I}$, so this case is eliminated.

Rewrite the $(s\bar{s}u)$, (stt) and (tst) equations:

$$(s\bar{s}u) \quad w_m^u = (u_u^1 + v_u^1) T^{s\bar{s}1} + v_t^1 T^{stm} T^{m\bar{s}u} + v_m^1 T^{smm} T^{m\bar{s}u} + u_t^1 T^{tsm} T^{smu} + u_m^1 T^{tsm} T^{smu} + w_s^m (T^{smu} + T^{m\bar{s}u}) \tag{70}$$

$$(stt) \quad w_s^1 = -v_m^1 (T^{sm1} + T^{smm} T^{mtt}) - v_u^1 T^{sum} T^{mtt} - w_s^m T^{mtt} - u_u^1 T^{utm} T^{smt} - u_m^1 T^{mtm} T^{smt} - w_t^m T^{smt} - (u_t^1 + v_t^1) T^{st1} \tag{71}$$

$$(tst) \quad w_s^1 = -u_m^1 (T^{ms1} + T^{mtm} T^{tmt}) - v_u^1 T^{sum} T^{smt} - v_m^1 T^{smm} T^{mst} - w_s^m T^{tmt} - u_u^1 T^{tmt} T^{mst} - w_t^m T^{mst} - (u_t^1 + v_t^1) T^{ts1}. \tag{72}$$

If $T^{smu} \neq 0$, we use (68) to get

$$w_t^m = -u_{\bar{t}}^1 T^{\bar{t}um} - u_m^1 T^{mtm}$$

for all $u \neq s, t$. We obtain $m-4$ relations on elements of $\mathcal{I}$, and thus this case is eliminated, and similarly we must have $T^{msu} = 0$. □

Remaining unassigned coefficients. Consider the (smm), (mtm) and (mmm) equations:

$$
\begin{aligned}
(smm)\quad & -(w^1_s + v^1_m T^{sm1})T^{1mm} - (h^t_s + z^t_s + v^1_s\delta_{st})T^{tmm} - (w^m_s + v^1_{\bar{s}})T^{mmm} - (u^1_m T^{m\bar{s}m})T^{s\bar{s}m}\\
& -(u^1_1 + u^1_u T^{umm} + u^1_m T^{mmm} + w^m_m)T^{smm} - (u^1_m + v^1_m)T^{sm1} + w^m_m T^{smm} = 0
\end{aligned}
$$

$$
\begin{aligned}
(msm)\quad & -(v^1_m T^{\bar{s}mm})T^{\bar{s}sm} - (v^1_1 + v^1_u T^{mum} + u^1_m T^{mmm} + w^m_m)T^{msm}\\
& -(w^1_s + u^1_m T^{ms1})T^{m1m} - (h^t_s + z^t_s + u^1_s\delta_{st})T^{mtm} - (w^m_s + u^1_{\bar{s}})T^{mmm}\\
& -(u^1_m + v^1_m)T^{mt1} + w^m_m T^{msm} = 0
\end{aligned}
$$

$$
\begin{aligned}
(mmm)\quad & -(v^1_\sigma T^{m\sigma 1} + w^1_m)T^{1mm} - (v^1_\sigma T^{m\sigma s} + w^s_m)T^{smm} - (v^1_1 + v^1_\sigma T^{m\sigma m} + w^m_m)T^{mmm}\\
& -(u^1_\sigma T^{\sigma m1} + w^1_m)T^{m1m} - (u^1_\sigma T^{\sigma ms} + w^s_m)T^{msm} - (u^1_1 + u^1_\sigma T^{\sigma mm} + w^m_m)T^{mmm}\\
& -(u^1_m + v^1_m)T^{mm1} + w^m_u T^{mmu} + w^m_m T^{mmm} = 0.
\end{aligned}
$$

First note that if some $T^{tmm} \neq 0$ the h^t_s are expressed in terms of elements of $\mathcal{I}$, which gives $m-2$ relations on the elements of $\mathcal{I}$, but then considering the $(m\bar{t}m)$ equation we also obtain v^1_m in terms of elements of $\mathcal{I}$, and considering the (mmm) equation we obtain $u^1_1 + v^1_1$ in terms of elements of $\mathcal{I}$. We conclude the coefficients T^{tmm} and by symmetry T^{mtm} are zero. With those coefficients zero, we also see $T^{mmm} = 0$ as otherwise the first two equations would express $v^1_{\bar{s}}, u^1_{\bar{s}}$ in terms of elements of $\mathcal{I}$.

Finally observe that the $(st1)$ equation is now

$$
u^s_m T^{mt1} + v^t_m T^{sm1} + w^1_m T^{stm} = 0,
$$

so if some $T^{mt1} \neq 0$ we obtain $m-2$ relations on the elements of $\mathcal{I}$. But then the (mtm) and (mmm) equations furnish two additional relations so we conclude.

Therefore all the unassigned coefficients are zero and the proof of Theorem A follows from the case by case calculations in the ensuing subsections.

Symmetry Lie algebras in the four cases.

Case **A1**: $T(\gamma^m)|_{\widehat{A}\otimes\widehat{B}}$ *is skew.* This is the case $T = T_{max,even,m}$. Write $m = 2q$. Let $\widehat{A} = \langle a_2, \dots, a_{m-1}\rangle$ and similarly for $\widehat{B}, \widehat{C}$. Recall that our normalizations give an identification $\widehat{A} \simeq \widehat{B}$. Let $2 \leq \xi \leq q$.

$$
\begin{aligned}
T_{maxeven,m} = & a_1\otimes b_1\otimes c_1 + a_1\otimes b_m\otimes c_m + a_m\otimes b_1\otimes c_m + \sum_{u=2}^{m-1}[a_1\otimes b_u\otimes c_u + a_u\otimes b_1\otimes c_u]\\
& + \sum_{\xi=2}^{q}[a_\xi\otimes b_{\xi+q-1}\otimes c_m - a_{\xi+q-1}\otimes b_\xi\otimes c_m].
\end{aligned}
$$

With blocking $(1, m-2, 1)\times(1, m-2, 1)$, we obtain:

$$
\widetilde{\mathfrak{g}}_{T_{maxeven,m}} =
$$

$$
\left\{
\begin{pmatrix} u^1_1 & x, y & u^1_m \\ 0 & u\mathrm{Id} + Z & \begin{smallmatrix} -f^t - \widetilde{y}^t \\ -g^t - \widetilde{x}^t \end{smallmatrix} \\ 0 & 0 & -(v^1_1 + w^m_m) \end{pmatrix},
\begin{pmatrix} v^1_1 & \widetilde{x}, \widetilde{y} & v^1_m \\ 0 & v\mathrm{Id} + Z & \begin{smallmatrix} -f^t - y^t \\ -g^t - x^t \end{smallmatrix} \\ 0 & 0 & -(u^1_1 + w^m_m) \end{pmatrix},
\begin{pmatrix} w^1_1 & 0 & 0 \\ \begin{smallmatrix} -y^t - \widetilde{y}^t \\ -x^t - \widetilde{x}^t \end{smallmatrix} & w\mathrm{Id} + Z & 0 \\ w^m_1 & f, g & w^m_m \end{pmatrix}
\;\middle|\;
\begin{matrix} f, g, x, y, \widetilde{x}, \widetilde{y} \in \mathbb{C}^{\frac{m-2}{2}}, \\ u^1_m + v^1_m + w^m_1 = 0, \\ u + v^1_1 + w = 0, \\ v + u^1_1 + w = 0, \\ u + v + w^m_m = 0, \\ u^1_1 + v^1_1 + w^1_1 = 0, \\ Z \in \mathfrak{sp}(m-2) \end{matrix}
\right\}.
$$

In particular $\dim \mathfrak{g}_{T_{maxeven,m}} = \frac{m^2}{2} + \frac{3m}{2} - 2$, and the solution space has dimension $\dim \mathfrak{h} + 3m - 1$, the largest allowed.

Case **A2***: $T(\gamma^m)$ is symmetric.* Using the diagonal form of Q, we obtain $T = T_{CW,m-2}$ in the following non-standard presentation:

$$T_{CW,m-2} = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_m \otimes c_m + a_m \otimes b_1 \otimes c_m + \sum_{u=2}^{m-1} [a_1 \otimes b_u \otimes c_u + a_u \otimes b_1 \otimes c_u + a_u \otimes b_u \otimes c_m].$$

With blocking $(1, m-2, 1) \times (1, m-2, 1)$, we obtain:

$$\widetilde{\mathfrak{g}}_{T_{CW,m-2}} = \left\{ \begin{pmatrix} u_1^1 & x & u_m^1 \\ 0 & u\mathrm{Id} + Z & -(y+z)^t \\ 0 & 0 & -(v_1^1 + w_m^m) \end{pmatrix}, \begin{pmatrix} v_1^1 & y & v_m^1 \\ 0 & v\mathrm{Id} + Z & -(x+z)^t \\ 0 & 0 & -(u_1^1 + w_m^m) \end{pmatrix}, \begin{pmatrix} w_1^1 & 0 & 0 \\ -(x+y)^t & w\mathrm{Id} + Z & 0 \\ w_1^m & z & w_m^m \end{pmatrix} \mid \begin{matrix} x, y, z \in \mathbb{C}^{m-2}, \\ u_1^1 + v_1^1 + w_1^1 = 0, \\ u + v_1^1 + w = 0, \\ v + u_1^1 + w = 0, \\ u + v + w_m^m = 0, \\ u_m^1 + v_m^1 + w_1^m = 0, \\ Z \in \mathfrak{so}(m-2) \end{matrix} \right\}.$$

In particular, $\dim \mathfrak{g}_{T_{CW,m-2}} = \frac{m^2}{2} + \frac{m}{2}$.

Case **A3***: skew part of $T(\gamma^m)|_{\hat{A} \otimes \hat{B}}$ has rank $m-2$ and symmetric part has rank one.* Write $m = 2q$.

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_m \otimes c_m + a_m \otimes b_1 \otimes c_m +$$

$$\sum_{u=2}^{m-1} [a_1 \otimes b_u \otimes c_u + a_u \otimes b_1 \otimes c_u] + a_2 \otimes b_2 \otimes c_m + \sum_{\xi=2}^{\frac{m}{2}+1} [a_\xi \otimes b_{\xi+q-1} \otimes c_m - a_{\xi+q-1} \otimes b_\xi \otimes c_m].$$

By Lemma 5.1 with $k = m-2$ even, in the case $s = 1$ and $\dim L = 1$, we have $\dim \mathfrak{h} = \binom{m-3}{2} + (m-2)$ and among the forms in I, a short calculation shows that only $u_1^1, v_1^1, w_m^m, w_2^1, w_m^1, w_m^t$ are nonzero (and independent). Consequently, we obtain $\dim \mathfrak{g}_T = [\binom{m-3}{2} + (m-2)] + [5 + (m-2)] - 2 = \frac{m^2}{2} - \frac{3m}{2} + 10$, which is too small for eligibility in Theorem A. This is in contrast to the other cases where the "nilpotent" part of $\mathfrak{g}_T$ is as large as naïvely possible.

Case **A4***: skew part of $T(\gamma^m)|_{\hat{A} \otimes \hat{B}}$ has rank $m-3$.* (Here $m = 2q+1$ is odd.) This is the case $T = T_{max,odd,skew,m}$. We obtain

(73)

$$T_{max,odd,skew,m} = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_m \otimes c_m + a_m \otimes b_1 \otimes c_m + \sum_{u=2}^{m-1} [a_1 \otimes b_u \otimes c_u + a_u \otimes b_1 \otimes c_u]$$

$$+ a_2 \otimes b_2 \otimes c_m + \sum_{\eta=3}^{q+1} [a_\eta \otimes b_{\eta+q-1} \otimes c_m - a_{\eta+q-1} \otimes b_\eta \otimes c_m].$$

Blocking $(1,1,m-3,1)\times(1,1,m-3,1)$, we have:

$$\widetilde{\mathfrak{g}}_{T_{max,odd,skew,m}} =$$

$$\left\{ \begin{pmatrix} u_1^1 & u_2^1 & x,y & u_m^1 \\ 0 & u_1^1 & 0 & -(v_2^1+w_2^m) \\ 0 & 0 & u_1^1\mathrm{Id}+Z & \substack{-f^t-\widetilde{y}^t \\ -g^t-\widetilde{x}^t} \\ 0 & 0 & 0 & u_1^1 \end{pmatrix}, \begin{pmatrix} v_1^1 & v_2^1 & \widetilde{x},\widetilde{y} & v_m^1 \\ 0 & v_1^1 & 0 & -(u_2^1+w_2^m) \\ 0 & 0 & v_1^1\mathrm{Id}+Z & \substack{-f^t-y^t \\ -g^t-x^t} \\ 0 & 0 & 0 & v_1^1 \end{pmatrix}, \begin{pmatrix} w_1^1 & 0 & 0 & 0 \\ -(u_2^1+v_2^1) & w_1^1 & 0 & 0 \\ \substack{-y^t-\widetilde{y}^t \\ -x^t-\widetilde{x}^t} & 0 & w_1^1\mathrm{Id}+Z & 0 \\ -(u_m^1+v_m^1) & w_2^m & f,g & w_1^1 \end{pmatrix} \mid \begin{matrix} f,g,x,y,\widetilde{x},\widetilde{y}\in\mathbb{C}^{\frac{m-3}{2}}, \\ u_1^1+v_1^1+w_1^1=0, \\ Z\in\mathfrak{sp}(m-3) \end{matrix} \right\}.$$

In particular, $\dim \mathfrak{g}_{T_{max,odd,skew,m}} = \frac{m^2}{2}+\frac{m}{2}$. Again the solution space has dimension $\dim\mathfrak{h}+3m-1$, the largest allowed. This concludes the proof of Theorem A.

8. Other tensors

We briefly describe the symmetry Lie algebras of other tensors used in the laser method and a related tensor.

Example 8.1 (Strassen's tensor)**.** The following is the first tensor that was used in the laser method: $T_{str,q} = \sum_{j=1}^q a_0\otimes b_j\otimes c_j + a_j\otimes b_0\otimes c_j \in \mathbb{C}^{q+1}\otimes\mathbb{C}^{q+1}\otimes\mathbb{C}^q$. Then, with blocking $(1,q)\times(1,q)$ in the first two matrices,

$$\widetilde{\mathfrak{g}}_{T_{str,q}} = \left\{ \lambda\mathrm{Id} + \begin{pmatrix} 0 & y \\ 0 & X \end{pmatrix}, \mu\mathrm{Id} + \begin{pmatrix} 0 & y \\ 0 & X \end{pmatrix}, \nu\mathrm{Id} + (-X^t) \mid X\in\mathfrak{gl}(q), x\in\mathbb{C}^q, \lambda+\mu+\nu=0 \right\}.$$

In particular, $\dim(\mathfrak{g}_{T_{str,q}}) = q^2+q$.

Example 8.2 (The small Coppersmith-Winograd tensor)**.** Another tensor used in the laser method is the small Coppersmith-Winograd tensor: $T_{cw,q} = \sum_{j=1}^q a_0\otimes b_j\otimes c_j + a_j\otimes b_0\otimes c_j + a_j\otimes b_j\otimes c_0 \in (\mathbb{C}^{q+1})^{\otimes 3}$. Then with blocking $(1,q)\times(1,q)$:

$$\widetilde{\mathfrak{g}}_{T_{cw,q}} = \left\{ \left(\begin{pmatrix} -\mu-\nu & 0 \\ 0 & \lambda\mathrm{Id}+X \end{pmatrix}, \begin{pmatrix} -\lambda-\nu & 0 \\ 0 & \mu\mathrm{Id}+X \end{pmatrix}, \begin{pmatrix} -\lambda-\mu & 0 \\ 0 & \nu\mathrm{Id}+X \end{pmatrix} \right) \mid \lambda,\mu,\nu\in\mathbb{C} X\in\mathfrak{so}(q) \right\}.$$

In particular $\dim\mathfrak{g}_{T_{cw,q}} = \binom{q}{2}+1$.

Example 8.3 (The tensor with maximal symmetry in §6 when $\overline{T}$ is symmetric)**.** Consider

$$T_{mcIsym,m} := a_1\otimes b_1\otimes c_1 + \sum_{\rho=2}^m (a_1\otimes b_\rho\otimes c_\rho + a_\rho\otimes b_1\otimes c_\rho + a_\rho\otimes b_\rho\otimes c_1).$$

Then with blocking $(1,m-1)\times(1,m-1)$,

$$\widetilde{\mathfrak{g}}_{T_{mcIsym,m}} = \left\{ \lambda\mathrm{Id} + \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}, \mu\mathrm{Id} + \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}, \nu\mathrm{Id} + \begin{pmatrix} & 0 \\ 0 & Z \end{pmatrix} \mid \begin{matrix} \lambda+\mu+\nu=0, \\ Z\in\mathfrak{so}_{m-1} \end{matrix} \right\}. \tag{74}$$

In particular, $\dim\mathfrak{g}_{T_{mcIsym,m}} = \binom{m-1}{2}$. This tensor is of interest because in [21], this tensor and the two Coppersmith-Winograd tensors were proven to be the unique 1-generic and maximally symmetrically compressible tensors.

9. Border rank bounds

A standard measure of complexity of a tensor T is its *border rank* $\underline{\mathbf{R}}(T)$, the smallest r such that $T \in \overline{GL_r^{\times 3} M_{\langle 1\rangle}^{\oplus r}}$. Strassen [25] showed that the exponent ω of matrix multiplication may be defined as the infimum over τ such that $\mathbf{R}(M_{\langle \mathbf{n}\rangle}) = O(\mathbf{n}^\tau)$, and Bini [5] showed one may use the border rank $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle})$ rather than the rank $\mathbf{R}(M_{\langle \mathbf{n}\rangle})$ in the definition. The tensor $T_{CW,m-2}$ has the minimal possible border rank m for any concise tensor, which is important for its use in proving upper bounds on ω.

Remark 9.1. The tensor of Proposition 3.1 satisfies $\mathbf{R}(a_1 \otimes (\sum_{j=1}^m b_j \otimes c_j)) = \underline{\mathbf{R}}(a_1 \otimes (\sum_{j=1}^m b_j \otimes c_j)) = m$.

Proposition 9.2. *Let $T \in A\otimes B\otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be such that in bases, where $\alpha^1, ..., \alpha^m$ is a basis of A^*, and*

$$T(A^*) = \begin{pmatrix} \alpha^1 & \alpha^2 & \alpha^3 & \cdots & \alpha^m \\ & \alpha^1 & & & \phi^2 \\ & & \ddots & & \phi^3 \\ & & & & \vdots \\ & & & & \phi^{m-1} \\ & & & & \alpha^1 \end{pmatrix},$$

where for at least one s_0, $\phi^{s_0} \notin \langle \alpha^{s_0}\rangle$. Then $\underline{\mathbf{R}}(T) \geq m+1$.

Proof. This is a straightforward application of Strassen's equations. Use $T(\alpha^1)$ to identify $B\otimes C$ with $\mathrm{End}(B)$. Write $\phi^{s_0} = \sum_\sigma c_\sigma \alpha^\sigma$ where say $c_{\sigma_1} \neq 0$ and $\sigma_1 \neq s_0$. Consider the commutator $[T(\alpha^{s_0}(a_{s_0})), T(\alpha^{\sigma_1}(a_{\sigma_1}))]$. It is a matrix whose $(1, m)$ entry is nonzero and thus T does not have minimal border rank. ☐

Corollary 9.3. *None of $T_{max,even,m} T_{max,odd,skew,m}, T_{max-1,odd,m}$ have minimal border rank m.*

Corollary 9.4. *Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be 1-generic and either symmetric or of minimal border rank. Then $\dim G_T \leq \binom{m+1}{2}$ with equality holding only for $T_{CW,m-2}$.*

Numerical computations using ALS methods (see [4]) indicate, at least for $m \leq 11$, that $\underline{\mathbf{R}}(T_{max,odd,skew,m}) \leq \frac{3m}{2} - \frac{1}{2}$ and for $m \leq 14$ that $\underline{\mathbf{R}}(T_{max,even,m}) \leq \frac{3m}{2} - 1$.

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