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Matrix product states and the quantum max-flow/min-cut conjectures

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In this note, we discuss the geometry of matrix product states with periodic boundary conditions and provide three infinite sequences of examples where the quantum max-flow is strictly less than the quantum min-cut. In the first, we fix the underlying graph to be a 4-cycle and verify a prediction of Hastings that inequality occurs for infinitely many bond dimensions. In the second, we generalize this result to a $2d$ -cycle. In the third, we show that the $2d$ -cycle with periodic boundary conditions gives inequality for all d when all bond dimensions equal two, namely, a gap of at least 2^{d-2} between the quantum max-flow and the quantum min-cut. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5026985>

I. INTRODUCTION

A *tensor network* associated to a graph is a way of constructing tensors in large spaces from smaller building-block tensors. From the perspective of algebraic geometry, tensor networks provide a natural way of constructing varieties of tensors. The use of graphs to study tensors dates back at least to Clifford in 1881 (see Ref. 1, Fig. 2.11.1). In applied mathematics and physics, tensor networks are versatile tools for efficiently approximating high-dimensional data, such as ground states of many-body quantum systems in condensed matter physics (see, e.g., Refs. 2–4).

In this paper, we focus exclusively on *translation-invariant matrix product states with periodic boundary conditions* as defined in Sec. I A. We use them to obtain three sequences of examples of tensor networks where the quantum max-flow (Sec. I B) is strictly less than the quantum min-cut (Sec. I C) for certain partitions. Our results disprove a natural conjecture and verify a numerical prediction from Ref. 5 (Sec. I D).

A. Matrix product states with periodic boundary conditions

In this paper, we work exclusively with tensor networks associated to the oriented cyclic graphs C_m , as in Fig. 1(a), and one building-block tensor T taken from $\mathbb{C}^N \otimes \mathbb{C}^n \otimes (\mathbb{C}^n)^*$, which we write as $A \otimes B \otimes B^*$.

One constructs tensors in $A^{\otimes m} = (\mathbb{C}^N)^{\otimes m}$ from this set-up as follows: For each vertex, we associate a copy of A to the external (physical) edge, a copy of B to the outgoing edge from the circle, and a copy of B^* to the incoming edge in the circle. We thus obtain a tensor $\Phi(T) \in A^{\otimes m}$ by

$$\Phi(T) = \text{contract}(T^{\otimes m}),$$

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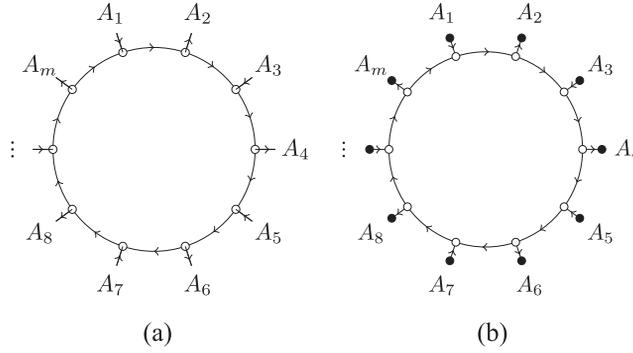


FIG. 1. (a) Cyclic graph C_m with m vertices and clockwise orientation. This graph is used for constructing a matrix product state. The m copies of the A vector space are labeled A_1, \dots, A_m . Arrows on the edges on the circle indicate B (outgoing) vs. B^* (incoming). Arrows on the external edges indicate partition into sources S (incoming) and sinks \bar{S} (outgoing). In our case, the quantum capacities are n for edges on the circle and N for external edges. (b) Extended graph \hat{C}_m obtained by adding terminal vertices (black) at the dangling ends of the external edges.

where we place a copy of T at each vertex, and then contract $B \otimes B^* \rightarrow \mathbb{C}$ along the edges on the circle. We will refer to the tensor $\Phi(T) \in A^{\otimes m}$ as the *tensor network state* associated to (C_m, T) . The dimensions N, n are known as the *bond dimensions*.

In physics, tensors such as $\Phi(T)$ are also known as *translation-invariant matrix product states with periodic boundary conditions*. In indices, if a_1, \dots, a_N is a basis of A , b_1, \dots, b_n is a basis of B with dual basis β^1, \dots, β^n , so we may write $T = \sum_{i,s,t} T^{is}_t a_i \otimes b_s \otimes \beta^t$, then

$$\Phi(T) = \sum T^{i_1 s_1}_{s_m} T^{i_2 s_2}_{s_1} \dots T^{i_m s_m}_{s_{m-1}} a_{i_1} \otimes \dots \otimes a_{i_m}. \tag{1}$$

If we think of each $T^i = (T^{is}_t)$ as an $n \times n$ matrix, we may write the $a_{i_1} \otimes \dots \otimes a_{i_m}$ coefficient of $\Phi(T)$ as

$$\text{trace}(T^{i_m} \dots T^{i_1}),$$

which explains the terminology “matrix product state.” “Periodic boundary conditions” refer to the fact that the graph is a cycle and not a linear graph, and “translation-invariant” means that the same tensor T is placed at each vertex. This implies a cyclic \mathbb{Z}_m -symmetry for $\Phi(T)$ that will play an important role in this paper.

Let $\text{TNS}(C_m, N, n) \subset A^{\otimes m}$ denote the set of all tensor network states $\Phi(T)$ associated to C_m , and let $\overline{\text{PTNS}}(C_m, N, n) \subset \mathbb{P}(A^{\otimes m})$ denote its Zariski closure in projective space. In other words, Φ defines a rational map

$$\Phi : \mathbb{P}(A \otimes B \otimes B^*) \dashrightarrow \mathbb{P}(A^{\otimes m})$$

(which we denote by the same symbol Φ), and $\overline{\text{PTNS}}(C_m, N, n)$ is the closure of its image. Note that the image is linearly degenerate, lying in the space of invariants $(A^{\otimes m})^{\mathbb{Z}_m}$, where \mathbb{Z}_m acts by cyclically permuting the vertices. Also note that the group $\text{GL}(B)$, the invertible linear maps $B \rightarrow B$, acts on B and B^* ; so $\text{PGL}(B) = \text{GL}(B)/\{\mathbb{C}^* \text{Id}\}$ acts on $\mathbb{P}(A \otimes B \otimes B^*)$, preserving the fibers of Φ . Thus the generic fiber of Φ contains $\text{PGL}(B)$.

B. Quantum max-flow

Now partition the external edges into two sets S, \bar{S} (“sources” and “sinks,” see Fig. 1). This induces a splitting $A^{\otimes m} = (\otimes_{s \in S} A_s) \otimes (\otimes_{t \in \bar{S}} A_t)$. Given $\Phi(T) \in A^{\otimes m}$, we get an induced linear map

$$\Phi(T)_{S, \bar{S}} : \bigotimes_{s \in S} A_s^* \rightarrow \bigotimes_{t \in \bar{S}} A_t$$

(called a *flattening* in the geometry literature). The *quantum max-flow* is defined as the maximal rank over all $\Phi(T)$ of this flattening⁶

$$\text{QMF}'(C_m, (S, \bar{S}), N, n) := \max_{T \in A \otimes B \otimes B^*} \text{rank}(\Phi(T)_{S, \bar{S}}).$$

The prime reminds us that we place the same tensor T at each vertex. When $N = n$, we suppress it from the notation and just write $\text{QMF}'(C_m, (S, \bar{S}), N)$.

C. Quantum min-cut

To define the quantum min-cut, it is useful to define an extended graph \widehat{C}_m by adding terminal vertices at the dangling ends of the external edges, as in Fig. 1(b). Now we may also think of the partition (S, \bar{S}) as a partition of the *terminals* into two sets. A *cut* in this situation is a partition of the vertices of \widehat{C}_m into two sets $(\mathcal{S}, \bar{\mathcal{S}})$, with $S \subseteq \mathcal{S}$ and $\bar{S} \subseteq \bar{\mathcal{S}}$. Define the *quantum capacity* of a cut $(\mathcal{S}, \bar{\mathcal{S}})$ by

$$\text{Qcap}(\mathcal{S}, \bar{\mathcal{S}}) := \prod_{v \in \mathcal{S}, w \in \bar{\mathcal{S}}, vw \in E} \text{qcap}(vw),$$

where the quantum capacity of an edge, $\text{qcap}(vw)$, is the dimension of the vector space associated to it. In our case, the quantum capacities are n for edges on the circle and N for external edges. Following Ref. 6, define the *quantum min-cut* by

$$\text{QMC}(C_m, (S, \bar{S}), N, n) := \min_{(\mathcal{S}, \bar{\mathcal{S}})} \text{Qcap}(\mathcal{S}, \bar{\mathcal{S}}),$$

where we minimize over all cuts $(\mathcal{S}, \bar{\mathcal{S}})$ for (S, \bar{S}) . Again we write $\text{QMC}(C_m, (S, \bar{S}), N)$, when $N = n$.

D. Quantum max-flow vs. quantum min-cut

In Ref. 6, the authors propose that tensor networks with physical edges divided into two sets can be viewed as “transporting” linear-algebraic quantities such as rank and entanglement, and are properly viewed as quantum analogs of graphs modeling flow networks. In the classical case, that is, for flow networks, it is well-known that the maximal flow passing from sources to sinks is equal to the minimum cut separating the sources from sinks—this is the famous *max-flow min-cut theorem*. In the quantum case, it is well-known and easy to see that

$$\text{QMF}'(C_m, (S, \bar{S}), (N, n)) \leq \text{QMC}(C_m, (S, \bar{S}), (N, n)) \tag{2}$$

and similarly for arbitrary graphs. Indeed, any cut $(\mathcal{S}, \bar{\mathcal{S}})$ induces a factorization of the linear map $\Phi(T)_{S, \bar{S}}$ through a vector space of dimension equal to the quantum capacity of the cut, which implies (2) at once.

In Ref. 6, the authors studied to what extent *equality* holds in the quantum case, prompted by the “quantum max-flow/min-cut conjecture” from Ref. 7, Conjecture C.1. Note that in the quantum case, the quantum min-cut is still straight-forward to compute (it can be readily reduced to computing a classical min-cut), but the quantum max-flow may be difficult to compute directly.

The original conjecture was vastly more general than the set-up here, but it in particular implied that, for all (S, \bar{S}) and N , $\text{QMF}'(C_m, (S, \bar{S}), N) = \text{QMC}(C_m, (S, \bar{S}), N)$. This was shown to be false in Ref. 6, namely, it was proved that $\text{QMF}'(C_4, (S, \bar{S}), 2) = 3 < 4 = \text{QMC}(C_4, (S, \bar{S}), 2)$, where $S = \{1, 3\}$, $\bar{S} = \{2, 4\}$. In Ref. 6, the cyclic graph C_4 was depicted as in Fig. 2(a), where the vertices in S are on the left and vertices in \bar{S} are on the right.

This raised the question whether weaker forms of a quantum max-flow/min-cut conjecture might be true, such as the following (stated for simplicity only for cyclic graphs):

Conjecture I.1. For any partition (S, \bar{S}) of the external edges, there exists N_0 such that, for all $N \geq N_0$,

$$\text{QMF}'(C_m, (S, \bar{S}), N) = \text{QMC}(C_m, (S, \bar{S}), N).$$

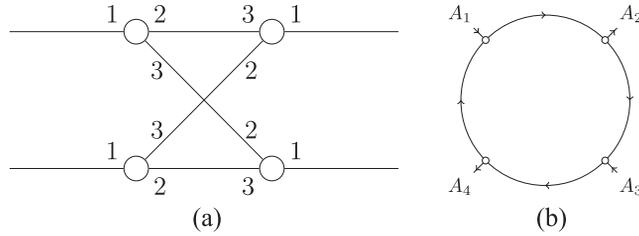


FIG. 2. (a) Cyclic graph C_4 as presented in Ref. 6. The labels 1, 2, 3 refers to the spaces A, B, B^* , respectively. (b) The same graph, but presented as in Fig. 1. The cyclic \mathbb{Z}_4 -symmetry is obvious.

We will be particularly interested in the case $m = 2d$, where S corresponds to the odd-labeled indices and \bar{S} corresponds to the even-labeled indices, as in Fig. 1(b). We will write $(S, \bar{S}) = (odd, even)$ accordingly. The example from Ref. 6 is of this form, see Fig. 2(b). Numerical evidence reported in Ref. 5 suggested a cyclic dependency of $\text{QMF}'(C_4, (odd, even), N)$ on $N \bmod 4$ in the situation of Fig. 2, and, therefore, that Conjecture I.1 is false. Our first result proves that this is indeed the following case:

Theorem I.2. For all N ,

$$\text{QMF}'(C_4, (odd, even), N) \leq \begin{cases} N^2 & \text{if } N \equiv 0, 1 \pmod{4} \\ N^2 - 1 & \text{if } N \equiv 2, 3 \pmod{4}, \end{cases}$$

while $\text{QMC}(C_4, (odd, even), N) = N^2$ for all N . Moreover, equality holds for all square bond dimensions $N = k^2$, so in an infinite number of cases where $N \equiv 0, 1 \pmod{4}$.

Numerical evidence suggests that our bound is tight.⁵ For $m > 4$, numerical evidence is much harder to obtain. Yet we provide a partial generalization of Theorem I.2 to higher cycles

Theorem I.3. For all $d \equiv 2 \pmod{4}$ and N ,

$$\text{QMF}'(C_{2d}, (odd, even), N) \leq \begin{cases} N^d & \text{if } N \not\equiv 3 \pmod{4} \\ N^d - 1 & \text{if } N \equiv 3 \pmod{4}, \end{cases}$$

while $\text{QMC}(C_{2d}, (odd, even), N) = N^d$ for all N . Again we have equality for all $N = k^2$.

We prove Theorems I.2 and I.3 in Sec. II.

Remark I.4. As suggested in Ref. 5, one might instead consider a weaker version of Conjecture I.1, where we only demand that equality holds for some (and hence infinitely many) $N > 1$. As evidence, Ref. 5 proved that for all $G, (S, \bar{S})$, as functions of N , we have $\text{QMF}'(G, (S, \bar{S}), N) = \text{QMC}(G, (S, \bar{S}), N)(1 - o(1))$ (cf. Refs. 8 and 9, where a similar result was proved in the scenario where we place different tensors at each site).

For $N = 2$, our bound in Theorem I.3 can be improved. This is shown by our next result, which gives an infinite sequence of graphs with constant bond dimension for which the quantum max-flow is strictly smaller than the quantum min-cut.

Theorem I.5. For all $d \geq 2$,

$$\text{QMF}'(C_{2d}, (odd, even), 2) \leq \frac{3}{4}2^d < 2^d = \text{QMC}(C_{2d}, (odd, even), 2).$$

Theorem I.5 puts the observation in Ref. 6 that $\text{QMF}'(C_6, (odd, even), 2) = 6 < 8$ into a general context. We prove Theorem I.5 in Sec. IV. Interestingly, the rank defect is of a *nonlinear* origin, unlike in our preceding theorems. Numerical calculations up to $d = 10$ suggest that our bound is tight.

Remark I.6. If $N = n = k^2$ for some integer k , then $\overline{TNS}(C_m, k^2, k^2)$ consists of the diagonal degenerations of the m -times iterated $k \times k$ -matrix multiplication tensor

$$\text{IMM}_k^m := \sum_{i_p=1}^k (a_1)_{i_2}^{i_1} \otimes (a_2)_{i_3}^{i_2} \otimes \cdots \otimes (a_{m-1})_{i_m}^{i_{m-1}} \otimes (a_m)_{i_1}^{i_m},$$

where $\{(a_p)_j^i\}_{ij=1,\dots,k}$ is a basis of the p th copy of A . The variety $TNS(C_m, k^2, k^2)$ is the closure of the image via Φ of the $GL(A)$ -orbit of the tensor $T = \sum a_j^i \otimes \beta_k^j \otimes b_i^k$, where $\{a_j^i\}_{ij=1,\dots,k}$ is a basis of A and $\{b_j^i\}_{ij=1,\dots,k}$ is a basis of B with dual basis $\{\beta_i^j\}_{ij=1,\dots,k}$. In fact, $T = \text{IMM}_k^3$ regarded as an element of $A \otimes B^* \otimes B$ and $\Phi(T) = \text{IMM}_k^m$.

The ranks of the flattenings of IMM_k^m are known (see, e.g., Refs. 10 and 11): when $m = 2d$ and $S = \{1, \dots, d\}$, $\bar{S} = \{d + 1, \dots, 2d\}$, the quantum min-cut is k^4 and indeed the flattening is of maximal rank k^4 . Similarly, in the case that $(S, \bar{S}) = (\text{odd}, \text{even})$, the quantum min-cut as well as the rank of the flattening is equal to k^{2d} . Thus

$$\text{QMF}'(C_{2d}, (\text{odd}, \text{even}), k^2) = k^{2d} = \text{QMC}(C_{2d}, (\text{odd}, \text{even}), k^2)$$

for all k . This shows the equality statements in Theorems I.2 and I.3.

E. Notation and conventions

A, B are complex vector spaces, respectively, of dimensions N, n . $GL(A)$ denotes the group of invertible linear maps $A \rightarrow A$, and \mathfrak{S}_d denotes the permutation group on d elements. We denote the elements of \mathbb{Z}_d , the cyclic group of order d , by $[k]$ for $k \in \mathbb{Z}$. We write $S^m A$ and $\Lambda^m A$ for the symmetric and antisymmetric subspaces of $A^{\otimes m}$, respectively.

II. PROOF OF THEOREMS I.2 AND I.3

We already showed in Remark I.6 that we have equality in all square dimensions $N = k^2$, so we only need to establish the upper bound. We first prove Theorem I.2, which rigorously establish the defects observed in Ref. 5, and in Theorem I.3.

A. Proof of Theorem I.2

It is clear from the cyclic symmetry of the tensor network in Fig. 2(b) that $\Phi(T) \in A^{\otimes 4}$ has a cyclic \mathbb{Z}_4 -symmetry, generated by $\pi = (1\ 2\ 3\ 4)$. Thus we need to understand the invariant subspace $(A^{\otimes 4})^{\mathbb{Z}_4}$.

For this, we order the tensor factors of $A^{\otimes 4}$ as $(A_1 \otimes A_3) \otimes (A_2 \otimes A_4)$, corresponding to the flattening $\Phi(T)_{\text{odd}, \text{even}}$ of interest. We claim that

$$(A^{\otimes 4})^{\mathbb{Z}_4} \subseteq S^2(S^2A) \oplus \Lambda^2(\Lambda^2A) \subseteq S^2A \otimes S^2A \oplus \Lambda^2A \otimes \Lambda^2A \subseteq (A_1 \otimes A_3) \otimes (A_2 \otimes A_4). \tag{3}$$

(In fact, the first inclusion is an equality.) To see this, recall that the \mathbb{Z}_4 -symmetry is generated by $\pi = (1\ 2\ 3\ 4)$. Since $\pi^2 = (1\ 3)(2\ 4)$, it is clear that $(A^{\otimes 4})^{\mathbb{Z}_4} \subseteq S^2A \otimes S^2A \oplus \Lambda^2A \otimes \Lambda^2A$. Now note that π acts block diagonally with respect to the direct sum. In fact, $\pi = (13)\tau$, where $\tau = (12)(34)$, so π and τ have the same action on $S^2A \otimes S^2A$, while π acts by $-\tau$ on $\Lambda^2A \otimes \Lambda^2A$. But τ interchanges $A_1 \otimes A_3$ with $A_2 \otimes A_4$, so it follows that the \mathbb{Z}_4 -invariant subspace lives in $S^2(S^2A) \oplus \Lambda^2(\Lambda^2A)$. This establishes (3).

Now, (3) implies that the linear map $\Phi(T)_{\text{odd}, \text{even}}$ is block diagonal, mapping the symmetric subspace of $A_1^* \otimes A_3^*$ to the symmetric subspace of $A_2 \otimes A_4$, and the anti-symmetric subspace to the anti-symmetric subspace. Moreover, the first block is given by a symmetric matrix, while the second block is given by a skew-symmetric matrix. Since the rank of a skew-symmetric matrix is always even, while $\dim \Lambda^2A = \frac{1}{2}N(N - 1)$ is even if and only if $N \equiv 0, 1 \pmod{4}$, we find that

$$\text{QMF}'(C_4, (\text{odd}, \text{even}), N) \leq \begin{cases} N^2 & \text{if } N \equiv 0, 1 \pmod{4} \\ N^2 - 1 & \text{if } N \equiv 2, 3 \pmod{4}. \end{cases} \tag{4}$$

(The $N = 2$ case will be re-proved geometrically in Sec. IV. It also follows by a direct computation, as was done in Ref. 6) □

B. Proof of Theorem I.3

Now consider a general graph C_{2d} with $d \equiv 2 \pmod{4}$. Again, $\Phi(T) \in A^{\otimes 2d}$ has a \mathbb{Z}_{2d} -symmetry, generated by $\pi = (1\ 2\ 3\ \dots\ 2d)$, so we focus on the \mathbb{Z}_{2d} -invariant subspace of $A^{\otimes 2d}$.

We first note that $\pi^2 = (1\ 3\ \dots\ 2d - 1)(2\ 4\ \dots\ 2d)$. Both $(1\ 3\ \dots\ 2d - 1)$ and $(2\ 4\ \dots\ 2d)$ are d -cycles, permuting the odd and even subsystems, respectively. We can decompose $A^{\otimes d} = \oplus_z V_z$ into the eigenspaces of such a d -cycle, where z runs over the d th roots of unity. Thus the invariance by π^2 implies that

$$(A^{\otimes 2d})^{\mathbb{Z}_{2d}} \subseteq \bigoplus_z V_z \otimes V_{\bar{z}},$$

where as before we use the odd-even ordering of tensor factors. Since d is even, $z = -1$ is a possible eigenvalue. Next, note that $\pi = (1\ 3\ \dots\ 2d - 1)\tau = \tau(2\ 4\ \dots\ 2d)$, where $\tau = (12)(34)\dots(2d - 1\ 2d)$ interchanges the odd and even subsystems. It follows that π acts by τ on $V_1 \otimes V_1$, by $-\tau$ on $V_{-1} \otimes V_{-1}$, and by $\begin{pmatrix} 0 & \bar{z}\tau \\ z\tau & 0 \end{pmatrix}$ on $V_z \otimes V_{\bar{z}} \oplus V_{\bar{z}} \otimes V_z$. In particular,

$$(A^{\otimes 2d})^{\mathbb{Z}_{2d}} \subseteq S^2(V_1) \oplus \Lambda^2(V_{-1}) \oplus \bigoplus_{z \neq \pm 1} V_z \otimes V_{\bar{z}}. \tag{5}$$

[For $d = 2$, the only eigenspaces are $V_1 = S^2A$ and $V_{-1} = \Lambda^2A$, so (5) reduces to (3)].

As before, (5) implies that $\Phi(T)_{\text{odd, even}}$ is block diagonal (e.g., with respect to the three direct summands) and the block that maps V_{-1}^* to V_{-1} is given by a skew-symmetric matrix. We now compute the dimension of the eigenspace V_{-1} .

Lemma II.1. *Let $d = 2(2a + 1)$. Then, $\dim V_{-1} = \frac{1}{2a+1} \sum_{b=1}^{2a+1} \binom{2a+1}{2}^{N^{\text{gcd}(2a+1,b)}}$. In particular, $\dim V_{-1}$ is odd if $N \equiv 3 \pmod{4}$.*

Proof. Consider the representation of \mathbb{Z}_d on $V = (\mathbb{C}^N)^{\otimes d}$ by cyclically shifting tensor factors. It is straightforward to evaluate its character $\chi: \mathbb{Z}_d \rightarrow \mathbb{C}$ in the standard product basis, $a_{i_{[1]}} \otimes \dots \otimes a_{i_{[d]}}$, where we label the indices by \mathbb{Z}_d rather than $\{1, \dots, d\}$ so that it is straightforward to implement the shift. For all $k \in \{1, \dots, d\}$,

$$\chi([k]) = \sum_{i_{[1]}, \dots, i_{[d]} \in \{1, \dots, N\}} \prod_{j=1}^d \delta_{i_{[j]}, i_{[j+k]}} = N^{|\mathbb{Z}_d/k\mathbb{Z}_d|} = N^{\text{gcd}(d,k)},$$

since we have one free index to choose per orbit of the shift by $[k]$. The dimension of the -1 eigenspace is the multiplicity of the sign representation in V , so given by the normalized inner product

$$\dim V_{-1} = \frac{1}{d} \sum_{k=1}^d (-1)^k \chi([k]) = \frac{1}{d} \sum_{k=1}^d (-1)^k N^{\text{gcd}(d,k)}. \tag{6}$$

Using $d = 2(2a + 1)$, we can calculate the numerator as

$$\begin{aligned} \sum_{k=1}^d (-1)^k N^{\text{gcd}(d,k)} &= \sum_{b=1}^{2a+1} N^{\text{gcd}(d,2b)} - N^{\text{gcd}(d,2b-1)} = \sum_{b=1}^{2a+1} N^{2\text{gcd}(2a+1,b)} - N^{\text{gcd}(2a+1,2b-1)} \\ &= \sum_{b=1}^{2a+1} N^{2\text{gcd}(2a+1,b)} - N^{\text{gcd}(2a+1,b)} = \sum_{b=1}^{2a+1} N^{\text{gcd}(2a+1,b)} (N^{\text{gcd}(2a+1,b)} - 1). \end{aligned}$$

In the third step, we substituted $2b - 1$ by b (since this defines a bijection of \mathbb{Z}_{2a+1} and $\text{gcd}(2a + 1, b)$ only depends on b modulo $2a + 1$, the sum is left unchanged). Thus the multiplicity (6) is given by

$$\dim V_{-1} = \frac{1}{d} \sum_{b=1}^{2a+1} N^{\gcd(2a+1,b)} (N^{\gcd(2a+1,b)} - 1) = \frac{1}{2a+1} \sum_{b=1}^{2a+1} \binom{N^{\gcd(2a+1,b)}}{2}.$$

This establishes the first claim. For the second, assume that $N \equiv 3 \pmod{4}$. In order to prove that $\dim V_{-1}$ is odd, it suffices to show that each binomial coefficient is odd. But indeed, since $2a + 1$ is odd, so is $\gcd(2a + 1, b)$. This implies that $N^{\gcd(2a+1,b)} \equiv 3 \pmod{4}$ for all b , which precisely ensures that the binomial coefficients are odd. \square

As before, Lemma II.1 implies that $\Phi(T)_{\text{odd,even}}$ has a rank defect if $N \equiv 3 \pmod{4}$; this establishes Theorem I.3. \square

Remark II.2. Numerical experiments suggest that there is a rank defect for all even d , not just for $d \equiv 2 \pmod{4}$.

III. THE VARIETY OF TENSOR NETWORK STATES $\overline{\mathbb{P}\text{TNS}}(C_m, N, n)$

In this section, we discuss some general features of the variety $\overline{\mathbb{P}\text{TNS}}(C_m, N, n)$. In particular, we show that, for $N, n \geq m$, the smallest linear subspace containing the variety is $(A^{\otimes m})^{\mathbb{Z}_m}$.

Let $v_m(\mathbb{P}A) := \{[p] \mid p = \ell^{\otimes m} \text{ for some } 0 \neq \ell \in A\} \subseteq \mathbb{P}(S^m A)$ denote the *Veronese variety* of m th powers of linear forms. For any variety $X \subseteq \mathbb{P}V$, let

$$\sigma_r(X) := \overline{\bigcup_{x_1, \dots, x_r \in X} \text{span}\{x_1, \dots, x_r\}}$$

denote the r th *secant variety* of X , so $\sigma_r(v_m(\mathbb{P}A)) \subseteq \mathbb{P}(S^m A)$ is the Zariski closure of the set of homogeneous polynomials of degree m in N variables that may be written as the sum of r m th powers of linear forms.

Observe that $\overline{\mathbb{P}\text{TNS}}(C_m, N, n)$ contains the variety $\sigma_n(v_m(\mathbb{P}A)) \subseteq \mathbb{P}(S^m A)$. Indeed, consider the tensor

$$T = a_1 \otimes b_1 \otimes \beta^1 + \dots + a_n \otimes b_n \otimes \beta^n,$$

where b_1, \dots, b_n is a basis of B , with dual basis β^1, \dots, β^n . Then (1) implies

$$\Phi(T) = a_1^{\otimes m} + \dots + a_n^{\otimes m}.$$

If the a_i are chosen as general points of A , this projectivizes to a general point of $\sigma_n(v_m(\mathbb{P}A))$, and by $\text{GL}(A)$ -invariance of the image, the whole variety must be contained in $\overline{\mathbb{P}\text{TNS}}(C_m, N, n)$.

Not every $\Phi(T)$ is contained in $S^m A$ (outside of the trivial case $n = 1$ which we exclude from consideration). For example, for $m = 2d$, consider the tensor

$$T = a_1 \otimes b_1 \otimes \beta^2 + a_2 \otimes b_2 \otimes \beta^1.$$

Then, with respect to the odd-even ordering of tensor factors,

$$\Phi(T) = a_1^{\otimes d} \otimes a_2^{\otimes d} + a_2^{\otimes d} \otimes a_1^{\otimes d}$$

which is not in $S^m A$.

Now assume that $N, n \geq m$ and consider the tensor

$$T = a_1 \otimes b_1 \otimes \beta^m + a_2 \otimes b_2 \otimes \beta^1 + \dots + a_{m-1} \otimes b_{m-1} \otimes \beta^{m-2} + a_m \otimes b_m \otimes \beta^{m-1}.$$

Then $\Phi(T)$ is the sum of the terms in the \mathbb{Z}_m -orbit of $a_1 \otimes a_2 \otimes \dots \otimes a_{m-1} \otimes a_m$. But such vectors span $(A^{\otimes m})^{\mathbb{Z}_m}$. In summary,

Proposition III.1. *If $N, n \geq m$, then $\text{span}(\overline{\mathbb{P}\text{TNS}}(C_m, N, n)) = (A^{\otimes m})^{\mathbb{Z}_m}$.*

It follows that rank violations that are not explained by the \mathbb{Z}_m -symmetry must be of a nonlinear origin. We will see such a phenomenon in Sec. IV.

IV. PROOF OF THEOREM I.5

In this section, we give a geometric construction of the relevant tensor network variety and prove Theorem I.5.

A. A geometric construction of $\overline{\mathbb{P}\text{TNS}}(C_{2d}, 2, 2)$

Let $A = B = \mathbb{C}^2$, let a_0, a_1 be a basis of A with dual basis α^0, α^1 of A^* , and b_0, b_1 be a basis of B with dual basis β^0, β^1 of B^* .

Consider the image under Φ of the line

$$L[\mu, \nu] := [\mu(a_0 \otimes b_0 \otimes \beta^0 + a_1 \otimes b_1 \otimes \beta^1) + \nu(a_0 \otimes b_0 \otimes \beta^1 + a_1 \otimes b_1 \otimes \beta^0)] \subseteq \mathbb{P}(A \otimes B \otimes B^*) = \mathbb{P}^7.$$

In Sec. III, we discussed the special cases $\nu = 0$ and $\mu = 0$. When $\nu = 0$, we obtain the point $[a_0^{\otimes 2d} + a_1^{\otimes 2d}]$, which is a general point of the variety $\sigma_2(v_{2d}(\mathbb{P}A)) \subseteq \mathbb{P}(S^{2d}A)$, which has dimension 3. When $\mu = 0$, we obtain $[a_0^{\otimes d} \otimes a_1^{\otimes d} + a_1^{\otimes d} \otimes a_0^{\otimes d}] \notin \mathbb{P}(S^{2d}A)$. Consider the closure of the union of the $\text{PGL}(A)$ -orbits of the points of $\Phi(L[\mu, \nu])$, that is $\overline{\text{PGL}(A) \cdot \Phi(L[\mu, \nu])}$: this is an irreducible projective variety of dimension 4 sitting inside $\overline{\mathbb{P}\text{TNS}}(C_{2d}, 2, 2)$ because $\dim \text{PGL}(A) = 3$ and the stabilizer of $[a_0^{\otimes 2d} + a_1^{\otimes 2d}]$ in $\text{PGL}(A)$ is finite.

But Φ is a rational map from $\mathbb{P}(A \otimes B \otimes B^*) = \mathbb{P}^7$ whose fibers have dimension at least $\dim \text{PGL}(B) = 3$. Thus the Zariski closure of its image $\overline{\mathbb{P}\text{TNS}}(C_{2d}, 2, 2)$ is an irreducible variety of dimension at most 4, so $\overline{\mathbb{P}\text{TNS}}(C_{2d}, 2, 2) = \overline{\text{PGL}(A) \cdot \Phi(L[\mu, \nu])}$. We summarize

Proposition IV.1. The variety $\overline{\mathbb{P}\text{TNS}}(C_{2d}, 2, 2)$ coincides with the variety $\overline{\text{PGL}(A) \cdot \Phi(L[\mu, \nu])}$.

We adopt the following notation: $\underline{\delta}$ denotes a $2d$ -tuple of elements in $\{0, 1\}$, and $\underline{\eta}$ and $\underline{\varepsilon}$ denote d -tuples of elements in $\{0, 1\}$. We use odd indices for the entries of $\underline{\eta}$ and even indices for the entries of $\underline{\varepsilon}$. We write $\underline{\eta} \smile \underline{\varepsilon}$ for the $2d$ -tuple obtained by interlacing $\underline{\eta}$ and $\underline{\varepsilon}$. Thus, if $\underline{\eta} = (\eta_1, \eta_3, \dots, \eta_{2d-1})$ and $\underline{\varepsilon} = (\varepsilon_2, \varepsilon_4, \dots, \varepsilon_{2d})$, then $\underline{\delta} = \underline{\eta} \smile \underline{\varepsilon} = (\eta_1, \varepsilon_2, \eta_3, \dots, \varepsilon_{2d})$. We write $a_{\underline{\delta}}$, etc., for the corresponding basis vectors. For example, the contraction of $a_{\underline{\eta} \smile \underline{\varepsilon}}$ by $\alpha^{\underline{\eta}}$ is $a_{\underline{\varepsilon}}$. All indices are to be read modulo $2d$.

For every $\underline{\delta} \in \{0, 1\}^{2d}$, define

$$\text{coeff}(\underline{\delta}) = \sum_{i=1}^{2d} \delta_i \boxplus \delta_{i+1}, \tag{7}$$

where \boxplus denotes the XOR operation, i.e., $\delta_i \boxplus \delta_{i+1} = 1$ if $\delta_i \neq \delta_{i+1}$ and $\delta_i \boxplus \delta_{i+1} = 0$ otherwise. The value $\text{coeff}(\underline{\delta})$ counts the number of changes between 0 and 1 that one observes cyclically reading $\underline{\delta}$. Directly from (1), we observe that $\text{coeff}(\underline{\delta})$ determines the coefficient of $\underline{\delta}$ in $\Phi(L(\mu, \nu))$. Explicitly,

$$\Phi(L(\mu, \nu)) = \sum_{\underline{\delta}} \mu^{2d-\text{coeff}(\underline{\delta})} \nu^{\text{coeff}(\underline{\delta})} a_{\underline{\delta}} = \sum_c \left[\mu^{2d-c} \nu^c \sum_{\text{coeff}(\underline{\delta})=c} a_{\underline{\delta}} \right]. \tag{8}$$

Moreover $\text{coeff}(\underline{\delta})$ is always even. See Fig. 3 for a schematic representation of the contribution of μ or ν in the contraction

Remark IV.2. The image of $L[\mu, \nu]$ under Φ is a rational normal curve of degree d , as opposed to the naïve $2d$. Indeed, since only even powers of μ and ν appear, one can reparametrize the image setting $\mu' = \mu^2, \nu' = \nu^2$, showing that it is a rational curve of degree at most d . Moreover, observe that

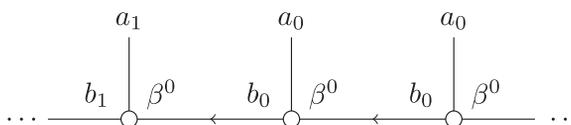


FIG. 3. This segment produces the monomial $\nu\mu\mu$.

for every even $c = 0, 2, \dots, 2d$ there exists a d -tuple $\underline{\delta}$ such that $\text{coeff}(\underline{\delta}) = c$ and the corresponding $a_{\underline{\delta}}$'s are linearly independent. This shows that $\Phi(L[\mu, \nu])$ spans a d -dimensional subspace of $\mathbb{P}(A^{\otimes 2d})$, which guarantees that it is a normal curve and that its degree is (at least) d .

B. Proof of Theorem I.5

Let $F := (\Phi(L(\mu, \nu)))_{\text{odd, even}}$ be the flattening of interest for some general choice of $[\mu, \nu] \in \mathbb{P}^1$. Since the rank of the flattening is $\text{GL}(A)$ -invariant, it is sufficient to prove $\text{rank}(F) \leq \frac{3}{4}2^d$, as $\Phi(L[\mu, \nu])$ is a general point of $\mathbb{P}\text{TNS}(C_{2d}, 2, 2)$.

We describe a set of elements in the kernel of F . Let $S \subseteq \text{odd} = \{1, 3, \dots, 2d - 1\}$ be a non-empty subset of even cardinality, $|S| = 2p$. Let $D_S = (\alpha^0 \otimes \alpha^1)^{\otimes p} - (\alpha^1 \otimes \alpha^0)^{\otimes p} \in (A^*)^{\otimes S}$ and let

$$K_S = D_S \otimes (\alpha^0 - \alpha^1)^{\otimes (d-2p)} \in (A^*)^{\otimes S} \otimes (A^*)^{\otimes (\text{odd} \setminus S)} = (A^*)^{\otimes \text{odd}}. \tag{9}$$

In this section, we will prove that $K_S \in \ker(F)$ for every S and that the K_S 's span a subspace of dimension at least 2^{d-2} . This provides $\text{rank}(F) \leq 2^d - 2^{d-2} = \frac{3}{4}2^d$ and therefore $\text{QMF}'(C_{2d}, (\text{odd}, \text{even}), 2) \leq \frac{3}{4}2^d$, establishing Theorem I.5.

For any fixed $S \subseteq \text{odd}$, the basis vectors $\alpha^{\underline{t}}$ in the support of K_S are labeled by d -tuples $\underline{\eta}$ that are in one-to-one correspondence with elements in $\{0, 1\} \times \{0, 1\}^{d-2p}$ as follows [here $\underline{\eta}'$ denote a $(d - 2p)$ -tuple]:

$$\begin{aligned} \{0, 1\} \times \{0, 1\}^{d-2p} &\rightarrow \{0, 1\}^S \times \{0, 1\}^{\text{odd} \setminus S} \cong \{0, 1\}^{\text{odd}}, \\ (0, \underline{\eta}') &\mapsto ((0, 1, \dots, 0, 1), \underline{\eta}'), \\ (1, \underline{\eta}') &\mapsto ((1, 0, \dots, 1, 0), \underline{\eta}'). \end{aligned}$$

We denote the elements of $\{0, 1\} \times \{0, 1\}^{d-2p}$ by $\underline{\eta} = (t, \underline{\eta}')$ and identify them with their image. The coefficient of $\alpha^{\underline{t}}$ in K_S is $\text{sign}(\underline{\eta}) := (-1)^{t+|\underline{t}'|}$, where $|\underline{\eta}'|$ is the sum of the entries of $\underline{\eta}'$, as can be readily seen from (9). Thus, $K_S = \sum_{\underline{\eta}} \text{sign}(\underline{\eta}) \alpha^{\underline{t}}$, and if we plug this into (8) then we obtain

$$F(K_S) = \sum_{\underline{\varepsilon}} \left(\sum_{\underline{\eta}} \text{sign}(\underline{\eta}) \mu^{2d - \text{coeff}(\underline{\eta} - \underline{\varepsilon})} \nu^{\text{coeff}(\underline{\eta} - \underline{\varepsilon})} \right) a_{\underline{\varepsilon}}.$$

Thus we can read off the following criterion:

Lemma IV.3. *The coefficient of $a_{\underline{\varepsilon}}$ in $F(K_S)$ is zero if there exists a permutation $\Theta_{\underline{\varepsilon}}$ of $\{0, 1\} \times \{0, 1\}^{d-2p}$ such that, for every $\underline{\eta}$, $\text{coeff}(\underline{\eta} - \underline{\varepsilon}) = \text{coeff}(\Theta_{\underline{\varepsilon}}(\underline{\eta}) - \underline{\varepsilon})$ while $\text{sign}(\Theta_{\underline{\varepsilon}}(\underline{\eta})) = -\text{sign}(\underline{\eta})$.*

We now show that such permutations always exist, which proves that each K_S is in the kernel.

Proposition IV.4. *For every subset $S \subseteq \text{odd} = \{1, 3, \dots, 2d - 1\}$ of even cardinality, and for every $\underline{\varepsilon} = (\varepsilon_2, \dots, \varepsilon_{2d})$, there exists a permutation $\Theta_{\underline{\varepsilon}}$ as in Lemma IV.3. As a consequence, each $K_S \in \ker(F)$.*

Proof. We distinguish two cases. First, assume that there exists an odd index $k \notin S$ such that $\varepsilon_{k-1} \neq \varepsilon_{k+1}$. Define an involution $\Theta_{\underline{\varepsilon}}$ that replaces η_k by its complement $\eta_k \boxplus 1$ (i.e., if $\eta_k = 0$, it is replaced by 1, and vice versa) while leaving all other elements the same. Then $\text{sign}(\Theta_{\underline{\varepsilon}}(\underline{\eta})) = -\text{sign}(\underline{\eta})$ for every $\underline{\eta}$. Moreover, $\text{coeff}(\Theta_{\underline{\varepsilon}}(\underline{\eta})) = \text{coeff}(\underline{\eta})$ because the only two terms in the summation (7) that involve η_k are

$$(\varepsilon_{k-1} \boxplus \eta_k) + (\eta_k \boxplus \varepsilon_{k+1}) = (0 \boxplus \eta_k) + (1 \boxplus \eta_k) = 1,$$

so their sum is independent of the value of η_k .

Now assume that $\varepsilon_{k-1} = \varepsilon_{k+1}$ for all odd $k \notin S$. Define an involution $\Theta_{\underline{\varepsilon}}(t, \underline{\eta}') = (t \boxplus 1, \underline{\eta}')$, i.e., every entry η_k for $k \in S$ is reversed while all other entries are unchanged. Again, it is clear

that $\text{sign}(\Theta_\varepsilon(\underline{\eta})) = -\text{sign}(\underline{\eta})$ for every $\underline{\eta}$. We now argue that $\text{coeff}(\Theta_\varepsilon(\underline{\eta})) = \text{coeff}(\underline{\eta})$. Let $S = \{k_1 < \dots < k_{2p}\}$. Then the only terms in the summation (7) that involve indices in S are

$$\sum_{j=1}^{2p} \left((\eta_{k_j} \boxplus \varepsilon_{k_{j+1}}) + (\varepsilon_{k_{j+1}-1} \boxplus \eta_{k_{j+1}}) \right), \tag{10}$$

where we set $k_{2p+1} := k_1$. Note that $\varepsilon_{k_{j+1}} = \varepsilon_{k_{j+1}-1}$ (for $k_{j+1} = k_j + 2$ this is trivial, otherwise use the assumption). Moreover, $\eta_{k_j} = 1 - \eta_{k_{j+1}}$, which remains true when we apply Θ_ε . As a consequence, applying Θ_ε only exchanges 0 and 1 in each summand of (10), and the total is preserved. \square

To conclude the Proof of Theorem I.5, it remains to show that $\mathbf{K} = \text{span}\{K_S : \emptyset \neq S \subseteq \text{odd}, |S| \text{ even}\}$ is of dimension at least 2^{d-2} . In fact, we will prove that $\mathbf{K}_{(1)} = \text{span}\{K_S : 1 \in S \subseteq \text{odd}, |S| \text{ even}\}$ has dimension equal to 2^{d-2} . Note that there are exactly 2^{d-2} subsets S of $\{1, 3, \dots, 2d-1\}$ with even cardinality that contain 1. Therefore, we need to show that the corresponding K_S are linearly independent. We start with the following lemma:

Lemma IV.5. We have $\mathbf{K}_{(1)} = \mathbf{K}_{(1,3)} \oplus \mathbf{K}_{(1,\bar{3})}$, where $\mathbf{K}_{(1,3)} := \text{span}\{K_S : 1, 3 \in S \subseteq \text{odd}, |S| \text{ even}\}$ and $\mathbf{K}_{(1,\bar{3})} := \text{span}\{K_S : 1 \in S \subseteq \text{odd}, 3 \notin S, |S| \text{ even}\}$. Furthermore, $\mathbf{K}_{(1,3)} = \mathbf{K}_{(1,3,5)} \oplus \mathbf{K}_{(1,3,\bar{5})}$, where $\mathbf{K}_{(1,3,5)} := \text{span}\{K_S : 1, 3, 5 \in S \subseteq \text{odd}, |S| \text{ even}\}$ and $\mathbf{K}_{(1,3,\bar{5})} := \text{span}\{K_S : 1, 3 \in S \subseteq \text{odd}, 5 \notin S, |S| \text{ even}\}$.

Proof. Clearly $\mathbf{K}_{(1,3)}$ and $\mathbf{K}_{(1,\bar{3})}$ generate $\mathbf{K}_{(1)}$. It remains to show that their intersection is trivial. Say it were not and let $v \in \mathbf{K}_{(1,3)} \cap \mathbf{K}_{(1,\bar{3})}$ be nonzero. Choose generic projections $A^* \rightarrow \mathbb{C}$ on the factors $5, \dots, 2d-1$ of $(A^*)^{\otimes \text{odd}}$, and consider the image of v in $A^* \otimes A^*$. On the one hand, since $v \in \mathbf{K}_{(1,3)}$, the image is of the form $x\alpha^0 \otimes \alpha^1 + y\alpha^1 \otimes \alpha^0$. By the genericity of the projections, we may assume that both x and y are nonzero so that the tensor has rank two. On the other hand, since $v \in \mathbf{K}_{(1,\bar{3})}$, the image is of the form of $(z\alpha^0 + w\alpha^1) \otimes (\alpha^1 - \alpha^0)$, a rank-one tensor. This is a contradiction. The second statement is proved analogously. \square

We now show that $\mathbf{K}_{(1)}$ has the desired dimension.

Lemma IV.6. For $d \geq 2$, we have that $\dim \mathbf{K}_{(1)} = 2^{d-2}$.

Proof. We will prove that $\dim \mathbf{K}_{(1)}^{(d)} = 2^{d-2}$ and $\dim \mathbf{K}_{(1,3)}^{(d)} = 2^{d-3}$ by induction on d . The base cases are as follows: $\mathbf{K}_{(1)}^{(2)} = \langle K_{\{1,3\}} \rangle$, $\mathbf{K}_{(1)}^{(3)} = \langle K_{\{1,3\}}, K_{\{1,5\}} \rangle$, and $\mathbf{K}_{(1,3)}^{(3)} = \langle K_{\{1,3\}} \rangle$ (recall we only consider nonempty subsets of even cardinality).

Now let $d \geq 4$. Using Lemma IV.5, we have

$$\mathbf{K}_{(1)}^{(d)} = \mathbf{K}_{(1,3)}^{(d)} \oplus \mathbf{K}_{(1,\bar{3})}^{(d)}, \quad \mathbf{K}_{(1,3)}^{(d)} = \mathbf{K}_{(1,3,5)}^{(d)} \oplus \mathbf{K}_{(1,3,\bar{5})}^{(d)}.$$

Note that $\mathbf{K}_{(1,\bar{3})}^{(d)} \subseteq (\alpha^0 - \alpha^1) \otimes (A^*)^{\otimes (\text{odd} \setminus \{3\})}$, so we see that $\mathbf{K}_{(1,\bar{3})}^{(d)} \cong \mathbf{K}_{(1)}^{(d-1)}$. Similarly, $\mathbf{K}_{(1,3,\bar{5})}^{(d)} \subseteq (\alpha^0 - \alpha^1) \otimes (A^*)^{\otimes (\text{odd} \setminus \{5\})}$ and $\mathbf{K}_{(1,3,\bar{5})}^{(d)} \cong \mathbf{K}_{(1,3)}^{(d-1)}$. Finally, note that $\mathbf{K}_{(1,3,5)}^{(d)} \cong \mathbf{K}_{(1)}^{(d-2)}$. Indeed, both spaces have the same number of generators; contraction with $a_0 \otimes a_1 + a_1 \otimes a_0 \in A^{\otimes \{1,3\}}$ maps each generator K_S onto a generator $K_{S'}$, where $S' = \{k-4 : k \in S, k \neq 1, 3\}$; all the latter are distinct and therefore linearly independent by the induction hypothesis. Thus,

$$\mathbf{K}_{(1)}^{(d)} \cong \mathbf{K}_{(1,3)}^{(d)} \oplus \mathbf{K}_{(1)}^{(d-1)}, \quad \mathbf{K}_{(1,3)}^{(d)} \cong \mathbf{K}_{(1)}^{(d-2)} \oplus \mathbf{K}_{(1,3)}^{(d-1)},$$

and so

$$\dim \mathbf{K}_{(1,3)}^{(d)} = 2^{(d-2)-2} + 2^{(d-1)-3} = 2^{d-3}, \quad \dim \mathbf{K}_{(1)}^{(d)} = 2^{d-3} + 2^{(d-1)-2} = 2^{d-2}$$

using the induction hypothesis. This concludes the proof. \square

Proposition IV.4 and Lemma IV.6 together establish Theorem I.5. \square

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