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Permanent v. determinant: An exponential lower bound assuming symmetry and a potential path towards Valiant's conjecture

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We initiate a study of determinantal representations with symmetry. We show that Grenet's determinantal representation for the permanent is optimal among determinantal representations equivariant with respect to left multiplication by permutation and diagonal matrices (roughly half the symmetry group of the permanent). We introduce a restricted model of computation, *equivariant determinantal complexity*, and prove an exponential separation of the permanent and the determinant in this model. This is the first exponential separation of the permanent from the determinant in any restricted model.

1. Introduction

Perhaps the most studied polynomial of all is the determinant:

$$\det_n(x) := \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^n_{\sigma(n)}, \tag{1}$$

a homogeneous polynomial of degree n in n^2 variables. Here $\mathfrak{S}_n$ denotes the group of permutations on n elements and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Despite its formula with $n!$ terms, $\det_n$ can be evaluated quickly, e.g., using Gaussian elimination, which exploits the large symmetry group of the determinant, e.g., $\det_n(x) = \det_n(AxB^{-1})$ for any $n\times n$ matrices A, B with determinant equal to one.

We will work exclusively over the complex numbers and with homogeneous polynomials, the latter restriction only for convenience. L. Valiant showed in [26] that given a homogeneous polynomial $P(y)$ in M

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variables, there exists an n and an affine linear map $\tilde{A} : \mathbb{C}^M \to \mathbb{C}^{n^2}$ such that $P = \det_n \circ \tilde{A}$. Such $\tilde{A}$ is called a *determinantal representation* of P. When $M = m^2$ and P is the permanent polynomial

$$\mathrm{perm}_m(y) := \sum_{\sigma \in \mathfrak{S}_m} y^1_{\sigma(1)} y^2_{\sigma(2)} \cdots y^m_{\sigma(m)}, \tag{2}$$

he showed that one can take $n = O(2^m)$. As an algebraic analog of the $\mathbf{P} \neq \mathbf{NP}$ conjecture, he also conjectured that one cannot do much better:

Conjecture 1.1 *(Valiant [27]). Let $n(m)$ be a function of m such that there exist affine linear maps $\tilde{A}_m : \mathbb{C}^{m^2} \to \mathbb{C}^{n(m)^2}$ satisfying*

$$\mathrm{perm}_m = \det_{n(m)} \circ \tilde{A}_m. \tag{3}$$

Then $n(m)$ grows faster than any polynomial in m.

To measure progress towards Conjecture 1.1, define $\mathrm{dc}(\mathrm{perm}_m)$ to be the smallest $n(m)$ such that there exists $\tilde{A}_m$ satisfying (3). The conjecture is that $\mathrm{dc}(\mathrm{perm}_m)$ grows faster than any polynomial in m. Lower bounds on $\mathrm{dc}(\mathrm{perm}_m)$ are: $\mathrm{dc}(\mathrm{perm}_m) > m$ (Marcus and Minc [16]), $\mathrm{dc}(\mathrm{perm}_m) > 1.06m$ (Von zur Gathen [29]), $\mathrm{dc}(\mathrm{perm}_m) > \sqrt{2}m - O(\sqrt{m})$ (Meshulam, reported in [29], and Cai [5]), with the current world record $\mathrm{dc}(\mathrm{perm}_m) \geq \frac{m^2}{2}$ [19] by Mignon and the second author. (Over $\mathbb{R}$, Yabe recently showed that $\mathrm{dc}_{\mathbb{R}}(\mathrm{perm}_m) \geq m^2 - 2m + 2$ [30], and in [6] Cai, Chen and Li extended the $\frac{m^2}{2}$ bound to arbitrary fields.)

Inspired by *Geometric Complexity Theory* (GCT) [20], we focus on the *symmetries* of $\det_n$ and perm_m. Let V be a complex vector space of dimension M, let $\mathrm{GL}(V)$ denote the group of invertible linear maps $V \to V$. For $P \in S^m V^*$, a homogeneous polynomial of degree m on V, let

$$G_P := \{g \in \mathrm{GL}(V) \mid P(g^{-1}y) = P(y) \quad \forall y \in V\}$$
$$\mathbb{G}_P := \{g \in \mathrm{GL}(V) \mid P(g^{-1}y) \in \mathbb{C}^* P(y) \quad \forall y \in V\}$$

denote the *symmetry group (resp. projective symmetry group)* of P. The function $\chi_P : \mathbb{G}_P \to \mathbb{C}^*$ defined by the equality $P(g^{-1}y) = \chi_P(g)P(y)$ is group homomorphism called the *character* of P. For example $\mathbb{G}_{\det_n} \simeq (\mathrm{GL}_n \times \mathrm{GL}_n)/\mathbb{C}^* \rtimes \mathbb{Z}_2$ [9], where the $\mathrm{GL}_n \times \mathrm{GL}_n$ invariance comes from $\det_n(AxB^{-1}) = (\det_n A \det_n B^{-1}) \det_n(x)$ and the $\mathbb{Z}_2$ is because $\det_n(x) = \det_n(x^T)$ where x^T is the transpose of the matrix x. Write $\tau : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{n^2}$ for the map $(A, B) \mapsto \{x \mapsto AxB^{-1}\}$. The character $\chi_{\det_n}$ satisfies $\chi_{\det_n} \circ \tau(A, B) = \det(A)\det(B)^{-1}$.

As observed in [20], the permanent (resp. determinant) is *characterized by its symmetries* and its degree in the sense that any polynomial $P \in S^m \mathbb{C}^{m^2 *}$ with a symmetry group G_P such that $G_P \supseteq G_{\mathrm{perm}_m}$ (resp. $G_P \supseteq G_{\det_m}$) is a scalar multiple of the permanent (resp. determinant). This property is the cornerstone of GCT. The program outlined in [20,21] is an approach to Valiant's conjecture based on the functions on GL_{n^2} that respect the symmetry group $G_{\det_n}$, i.e., are invariant under the action of $G_{\det_n}$.

The interest in considering $\mathbb{G}_P$ instead of G_P is that if P is characterized by G_P among homogeneous polynomials of the same degree, then it is characterized by the pair $(\mathbb{G}_P, \chi_P)$ among all polynomials. This will be useful, since *a priori*, $\det_n \circ \tilde{A}$ need not be homogeneous.

Guided by the principles of GCT, we ask:

What are the $\tilde{A}$ that respect the symmetry group of the permanent?

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To make this question precise, let $\tilde{A}$ be a determinantal representation of $P \in S^m V^*$. Note that if $P = \mathrm{perm}_m$ then V is the space $\mathcal{M}_m(\mathbb{C})$ of $m \times m$ matrices. Write the affine linear map $\tilde{A}$ as $\tilde{A} = \Lambda + A$ where $\Lambda \in \mathcal{M}_n(\mathbb{C})$ is a fixed matrix and $A : V \to \mathcal{M}_n(\mathbb{C})$ is linear.

The subgroup $\mathbb{G}_{\det_n} \subset \mathrm{GL}_{n^2}$ satisfies

$$\mathbb{G}_{\det_n} \simeq (\mathrm{GL}_n \times \mathrm{GL}_n)/\mathbb{C}^* \rtimes \mathbb{Z}_2. \tag{4}$$

Definition 1.2. Let $\tilde{A} : V \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a determinantal representation of $P \in S^m V^*$. Define

$$\mathbb{G}_A = \{g \in \mathbb{G}_{\det_n} \mid g \cdot \Lambda = \Lambda \text{ and } g \cdot A(V) = A(V)\},$$

the *symmetry group of the determinantal representation* $\tilde{A}$ of P.

The group $\mathbb{G}_A$ comes with a representation $\rho_A : \mathbb{G}_A \longrightarrow \mathrm{GL}(A(V))$ obtained by restricting the action to $A(V)$. We assume that P cannot be expressed using $\dim (V) - 1$ variables, i.e., that $P \notin S^m V'$ for any hyperplane $V' \subset V^*$. Then $A : V \longrightarrow A(V)$ is bijective. Let $A^{-1} : A(V) \longrightarrow V$ denote its inverse. Set

$$\begin{aligned} \bar{\rho}_A : \mathbb{G}_A &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto A^{-1} \circ \rho_A(g) \circ A. \end{aligned} \tag{5}$$

Definition 1.3. We say $\tilde{A}$ is an *equivariant representation* of P if (5) surjects onto $\mathbb{G}_P$.

If G is a subgroup of $\mathbb{G}_P$, we say that $\tilde{A}$ *is equivariant with respect to* G if G is contained in the image of $\bar{\rho}_A$.

Example 1.4. Let $Q = \sum_{j=1}^{M} z_j^2 \in S^2\mathbb{C}^{M*}$ be a nondegenerate quadric. Then $\mathbb{G}_Q = \mathbb{C}^* \times O(M)$ where $O(M) = \{B \in GL_M \mid B^{-1} = B^T\}$ is the orthogonal group, as for such B, $B \cdot Q = \sum_{i,j,k} B_{i,j} B_{k,j} z_i z_k = \sum_{ij} \delta_{ij} z_i z_j = Q$. Consider the determinantal representation

$$Q = \det_{M+1} \begin{pmatrix} 0 & -z_1 & \cdots & -z_M \\ z_1 & 1 & & \\ \vdots & & \ddots & \\ z_M & & & 1 \end{pmatrix}. \tag{6}$$

For $(\lambda, B) \in \mathbb{G}_Q$, define an action on $Z \in \mathcal{M}_{M+1}(\mathbb{C})$ by

$$Z \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} Z \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & B \end{pmatrix}^{-1}.$$

Write

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} \qquad \text{so} \qquad \tilde{A} = \begin{pmatrix} 0 & -X^T \\ X & \mathrm{Id}_M \end{pmatrix}.$$

The relation $B^{-1} = B^T$ implies

$$\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 0 & -X^T \\ X & \mathrm{Id}_M \end{pmatrix} \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -(\lambda BX)^T \\ \lambda BX & \mathrm{Id}_M \end{pmatrix}.$$

Taking $\det_{M+1}$ on both sides gives

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$$\lambda^2 Q(X) = (\lambda, B) \cdot Q(X).$$

Thus $\tilde{A}$ is an equivariant representation of Q.

Roughly speaking, $\tilde{A}$ is an equivariant representation of P if for any $g \in \mathbb{G}_P$, one can recover the fact that $P(g^{-1}x) = \chi_P(g)P(x)$ just by applying Gaussian elimination and $\det(Z^T) = \det(Z)$ to $\tilde{A}$.

Definition 1.5. For $P \in S^m V^*$, define the *equivariant determinantal complexity* of P, denoted $\mathrm{edc}(P)$, to be the smallest n such that there is an equivariant determinantal representation of P.

Of course $\mathrm{edc}(P) \geq \mathrm{dc}(P)$. If P is generic, or $P = \det_m$, then $\mathrm{edc}(P) = \mathrm{dc}(P)$. Our main result is that $\mathrm{edc}(\mathrm{perm}_m)$ is exponential in m.

2. Results

2.1. Main theorem

Theorem 2.1. *Let $m \geq 3$. Then* $\mathrm{edc}(\mathrm{perm}_m) = \binom{2m}{m} - 1 \sim 4^m$.

There are several instances in complexity theory where an optimal algorithm partially respects symmetry, e.g. Strassen's algorithm for 2×2 matrix multiplication is invariant under the $\mathbb{Z}_3$-symmetry of the matrix multiplication operator (in fact more, see [4]).

For the purposes of Valiant's conjecture, we ask the weaker question:

Question 2.2. Does there exist a polynomial $e(d)$ such that $\mathrm{edc}(\mathrm{perm}_m) \leq e(\mathrm{dc}(\mathrm{perm}_m))$?

Theorem 2.1 implies:

Corollary 2.3. *If the answer to Question 2.2 is affirmative, then Conjecture 1.1 is true.*

We have no opinion as to what the answer to Question 2.2 should be, but as it provides a new potential path to proving Valiant's conjecture, it merits further investigation. Note that Question 2.2 is a *flip* in the terminology of [22], since a positive answer is an existence result. It fits into the more general question: *When an object has symmetry, does it admit an optimal expression that preserves its symmetry?*

Example 2.4. Let $T \in W^{\otimes d}$ be a symmetric tensor, i.e. $T \in S^d W \subset W^{\otimes d}$. Say T can be written as a sum of r rank one tensors, then P. Comon conjectures [7] that it can be written as a sum of r rank one symmetric tensors.

Example 2.5 *(Optimal Waring decompositions).* The optimal Waring decomposition of $x_1 \cdots x_n$, dating back at least to [8] and proved to be optimal in [24] is

$$x_1 \cdots x_n = \frac{1}{2^{n-1} n!} \sum_{\substack{\epsilon \in \{-1,1\}^n \\ \epsilon_1 = 1}} \Big(\sum_{j=1}^{n} \epsilon_j x_j \Big)^n \Pi_{i=1}^{n} \epsilon_i, \tag{7}$$

a sum with 2^{n-1} terms. This decomposition has a transparent $\mathfrak{S}_{n-1}$-symmetry. As pointed out by H. Lee [15], when n is odd it even has an $\mathfrak{S}_n$-symmetry, and it has been known since 1934 [18] that by doubling the size one obtains an $\mathfrak{S}_n$-symmetry for all n. These decompositions do not preserve the action of the torus T^{SL_n} of diagonal matrices with determinant one.

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The optimal Waring decomposition of the permanent is not known, but it is known to be of size greater than $\binom{n}{\lfloor n/2\rfloor}^2 \sim 4^n/\sqrt{n}$. The Ryser–Glynn formula [10] is

$$\mathrm{perm}_n(x) = 2^{-n+1} \sum_{\substack{\epsilon\in\{-1,1\}^n\\ \epsilon_1=1}} \prod_{1\le i\le n} \sum_{1\le j\le n} \epsilon_i\epsilon_j x_{i,j}, \tag{8}$$

the outer sum taken over n-tuples $(\epsilon_1 = 1, \epsilon_2, \dots, \epsilon_n)$. This $\mathfrak{S}_{n-1}\times\mathfrak{S}_{n-1}$-invariant formula can also be made $\mathfrak{S}_n\times\mathfrak{S}_n$-invariant by enlarging it by a factor of 4, to get a $\mathfrak{S}_n\times\mathfrak{S}_n$ homogeneous depth three formula that is within a factor of four of the best known. Then expanding each monomial above, using the $\mathfrak{S}_n$-invariant decomposition of $x_1\cdots x_n$, one gets a $\mathfrak{S}_n\times\mathfrak{S}_n$-Waring expression within a factor of $O(\sqrt{n})$ of the lower bound.

Example 2.6. Examples regarding equivariant representations of $\mathfrak{S}_N$-invariant functions from the Boolean world give inconclusive indications regarding Question 2.2.

The MOD_m-degree of a Boolean function $f(x_1,\dots,x_N)$ is the smallest degree of any polynomial $P\in\mathbb{Z}[x_1,\dots,x_N]$ such that $f(x)=0$ if and only if $P(x)=0$ for all $x\in\{0,1\}^N$. The known upper bound for the MOD_m-degree of the Boolean OR function ($OR(x_1,\dots,x_N)=1$ if any $x_j=1$ and is zero if all $x_j=0$) is attained by symmetric polynomials [3]. Moreover in [3] it is also shown that this bound cannot be improved with symmetric polynomials, and it is far from the known lower bound.

The boolean majority function $MAJ(x_1,\dots,x_N)$ takes on 1 if at least half the $x_j=1$ and zero otherwise. The best monotone Boolean formula for MAJ [28] is polynomial in N and attained using random functions, and it is expected that the only symmetric monotone formula for majority is the trivial one, disjunction of all $\frac{n}{2}$-size subsets (or its dual), which is of exponential size.

Question 2.7. Does every polynomial P that is determined by its symmetry group admit an equivariant determinantal representation? For those P that do, how much larger must such a determinantal representation be from the size of a minimal one?

2.2. Grenet's formulas

The starting point of our investigations was the result in [1] that $\mathrm{dc}(\mathrm{perm}_3)=7$, in particular Grenet's determinantal expression [11] for perm_3:

$$\mathrm{perm}_3(y) = \det{}_7\begin{pmatrix} 0 & 0 & 0 & 0 & y^3_3 & y^3_2 & y^3_1\\ y^1_1 & 1 & & & & & \\ y^1_2 & & 1 & & & & \\ y^1_3 & & & 1 & & & \\ & y^2_2 & y^2_1 & 0 & 1 & & \\ & y^2_3 & 0 & y^2_1 & & 1 & \\ & 0 & y^2_3 & y^2_2 & & & 1 \end{pmatrix}, \tag{9}$$

is optimal. We sought to understand (9) from a geometric perspective. A first observation is that it, and more generally Grenet's representation for perm_m as a determinant of size 2^m-1 is equivariant with respect to about half the symmetries of the permanent. In particular, the optimal expression for perm_3 is equivariant with respect to about half its symmetries.

To explain this observation, introduce the following notation. Write $\mathcal{M}_m(\mathbb{C}) = \mathrm{Hom}(F,E) = F^*\otimes E$, where $E,F=\mathbb{C}^m$. This distinction of the two copies of $\mathbb{C}^m$ clarifies the action of the group $\mathrm{GL}(E)\times\mathrm{GL}(F)$ on $\mathrm{Hom}(F,E)$. This action is $(A,B)\cdot x = AxB^{-1}$, for any $x\in\mathrm{Hom}(F,E)$ and $(A,B)\in\mathrm{GL}(E)\times\mathrm{GL}(F)$. Let

$T^{\mathrm{GL}(E)} \subset \mathrm{GL}(E)$ consist of the diagonal matrices and let $N(T^{\mathrm{GL}(E)}) = T^{\mathrm{GL}(E)} \rtimes \mathfrak{S}_m \subset \mathrm{GL}(E)$ be its normalizer, where $\mathfrak{S}_m$ denotes the group of permutations on m elements. Similarly for $T^{\mathrm{GL}(F)}$ and $N(T^{\mathrm{GL}(F)})$. Then $\mathbb{G}_{\mathrm{perm}_m} \simeq [(N(T^{\mathrm{GL}(E)}) \times N(T^{\mathrm{GL}(F)}))/\mathbb{C}^*] \rtimes \mathbb{Z}_2$, where the embedding of $(N(T^{\mathrm{GL}(E)}) \times N(T^{\mathrm{GL}(F)}))/\mathbb{C}^*$ in $\mathrm{GL}(\mathrm{Hom}(F, E))$ is given by the action above and the term $\mathbb{Z}_2$ corresponds to transposition. This was first shown in [17].

The following refinement of Theorem 2.1 asserts that to get an exponential lower bound it is sufficient to respect about half the symmetries of the permanent.

Theorem 2.8. *Let $m \geq 3$. Let $\tilde{A}_m : \mathcal{M}_m(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a determinantal representation of perm_m that is equivariant with respect to $N(T^{\mathrm{GL}(E)})$. Then $n \geq 2^m - 1$.*

Moreover, Grenet's determinantal representation of perm_m is equivariant with respect to $N(T^{\mathrm{GL}(E)})$ and has size $2^m - 1$.

We now explain Grenet's expressions from a representation-theoretic perspective. Let $[m] := \{1, \dots, m\}$ and let $k \in [m]$. Note that $S^k E$ is an irreducible $\mathrm{GL}(E)$-module but it is not irreducible as an $N(T^{\mathrm{GL}(E)})$-module. Let $e_1, \dots, e_m$ be a basis of E, and let $(S^k E)_{reg}$ denote the span of $\prod_{i \in I} e_i$, for $I \subset [m]$ of cardinality k (the space spanned by the square-free monomials, also known as the space of *regular* weights): $(S^k E)_{reg}$ is an irreducible $N(T^{\mathrm{GL}(E)})$-submodule of $S^k E$. Moreover, there exists a unique $N(T^{\mathrm{GL}(E)})$-equivariant projection π_k from $S^k E$ to $(S^k E)_{reg}$.

For $v \in E$, define $s_k(v) : (S^k E)_{reg} \to (S^{k+1} E)_{reg}$ to be multiplication by v followed by π_{k+1}. Alternatively, $(S^{k+1} E)_{reg}$ is an $N(T^{\mathrm{GL}(E)})$-submodule of $E \otimes (S^k E)_{reg}$, and $s_k : E \to (S^k E)^*_{reg} \otimes (S^{k+1} E)_{reg}$ is the unique $N(T^{\mathrm{GL}(E)})$-equivariant inclusion. Let $\mathrm{Id}_W : W \to W$ denote the identity map on the vector space W. Fix a basis $f_1, \dots, f_m$ of F^*.

Proposition 2.9. *The following is Grenet's determinantal representation of perm_m. Let $\mathbb{C}^n = \bigoplus_{k=0}^{m-1} (S^k E)_{reg}$, so $n = 2^m - 1$, and identify $S^0 E \simeq (S^m E)_{reg}$. Set*

$$\Lambda_0 = \sum_{k=1}^{m-1} \mathrm{Id}_{(S^k E)_{reg}}$$

and define

$$\tilde{A} = \Lambda_0 + \sum_{k=0}^{m-1} s_k(E) \otimes f_{k+1}. \tag{10}$$

Then $(-1)^{m+1} \mathrm{perm}_m = \det_n \circ \tilde{A}$. To obtain the permanent exactly, replace $\mathrm{Id}_{(S^1 E)_{reg}}$ by $(-1)^{m+1} \mathrm{Id}_{(S^1 E)_{reg}}$ in the formula for Λ_0.

In bases respecting the block decomposition induced from the direct sum, the linear part, other than the last term which lies in the upper right block, lies just below the diagonal blocks, and all blocks other than the upper right block and the diagonal and sub-diagonal blocks, are zero.

Moreover $N(T^{\mathrm{GL}(E)}) \subseteq \bar{\rho}_A(\mathbb{G}_A)$.

2.3. An equivariant representation of the permanent

We now give a minimal equivariant determinantal representation of perm_m. By Theorem 2.1, its size is $\binom{2m}{m} - 1$. For $e \otimes f \in E \otimes F^*$, let $S_k(e \otimes f) : (S^k E)_{reg} \otimes (S^k F^*)_{reg} \to (S^{k+1} E)_{reg} \otimes (S^{k+1} F^*)_{reg}$ be multiplication by e on the first factor and f on the second followed by projection into $(S^{k+1} E)_{reg} \otimes (S^{k+1} F^*)_{reg}$. Equivalently,

$$S_k : (E \otimes F^*) \to ((S^k E)_{reg} \otimes (S^k F^*)_{reg})^* \otimes (S^{k+1} E)_{reg} \otimes (S^{k+1} F^*)_{reg}$$

is the unique $N(T^{\mathrm{SL}(E)}) \times N(T^{\mathrm{SL}(F)})$ equivariant inclusion.

Proposition 2.10. *The following is an equivariant determinantal representation of* perm_m*: Let* $\mathbb{C}^n = \oplus_{k=0}^{m-1} (S^k E)_{reg} \otimes (S^k F^*)_{reg}$*, so* $n = \binom{2m}{m} - 1 \sim 4^m$*. Fix a linear isomorphism* $S^0 E \otimes S^0 F^* \simeq (S^m E)_{reg} \otimes (S^m F^*)_{reg}$*. Set*

$$\Lambda_0 = \sum_{k=1}^{m-1} \mathrm{Id}_{(S^k E)_{reg} \otimes (S^k F^*)_{reg}}$$

and define

$$\tilde{A} = (m!)^{\frac{-1}{n-m}} \Lambda_0 + \sum_{k=0}^{m-1} S_k(E \otimes F). \tag{11}$$

Then $(-1)^{m+1} \mathrm{perm}_m = \det_n \circ \tilde{A}$*. In bases respecting the block structure induced by the direct sum, except for* S_{m-1}*, which lies in the upper right hand block, the linear part lies just below the diagonal block.*

2.4. Determinantal representations of quadrics

It will be instructive to examine other polynomials determined by their symmetry groups. Perhaps the simplest such is a nondegenerate quadratic form.

Let $Q = \sum_{j=1}^{s} x_j y_j$ be a non-degenerate quadratic form in $2s$ variables. The polynomial Q is characterized by its symmetries. By [19], if $s \geq 3$, the smallest determinantal representation of Q is of size $s+1$:

$$\tilde{A} = \begin{pmatrix} 0 & -x_1 & \cdots & -x_s \\ y^1 & 1 & & \\ \vdots & & \ddots & \\ y^s & & & 1 \end{pmatrix}. \tag{12}$$

As described in §4, this representation is equivariant with respect to about “half” the symmetry group $\mathbb{G}_Q$. We show in §4 that there is no size $s+1$ $\mathbb{G}_Q$-equivariant determinantal representation. However, Example 1.4 shows there is a size $2s+1$ determinantal representation respecting $\mathbb{G}_Q$.

Proposition 2.11. *Let* $Q \in S^2 \mathbb{C}^{M*}$ *be a nondegenerate quadratic form, that is, a homogeneous polynomial of degree 2. Then*

$$\mathrm{edc}(Q) = M + 1.$$

2.5. Determinantal representations of the determinant

Although it may appear strange at first, one can ask for determinantal representations of $\det_m$. In this case, to get an interesting lower bound, we add a regularity condition motivated by Lemma 3.2:

Definition 2.12. Let $P \in S^m V^*$. A determinantal representation $\tilde{A} : V \longrightarrow \mathcal{M}_n(\mathbb{C})$ is said to be *regular* if $\tilde{A}(0)$ has rank $n-1$.

Call the minimal size of a regular determinantal representation of P the *regular determinantal complexity of* P and denote it by $\mathrm{rdc}(P)$. Let $\mathrm{erdc}(P)$ denote the minimal size of a regular equivariant determinantal representation of P.

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Any determinantal representation of perm_m or a smooth quadric is regular, see §3.3. In contrast, the trivial determinantal representation of $\det_m$ is not regular; but this representation is equivariant so $\mathrm{edc}(\det_m) = m$.

Theorem 2.13. $\mathrm{erdc}(\det_m) = \binom{2m}{m} - 1 \sim 4^m$.

As in the case of the permanent, we can get an exponential lower bound using only about half the symmetries of the determinant.

Theorem 2.14. *Let $\tilde{A}_m : \mathcal{M}_m(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a regular determinantal representation of $\det_m$ that is equivariant with respect to $\mathrm{GL}(E)$. Then $n \geq 2^m - 1$.*

Moreover, there exists a regular determinantal representation of $\det_m$ that is equivariant with respect to $\mathrm{GL}(E)$ of size $2^m - 1$.

Remark 2.15. Normally when one obtains the same lower bound for the determinant as the permanent in some model it is discouraging for the model. However here there is an important difference due to the imposition of regularity for the determinant. We discuss this in further below Question 2.18.

We now introduce notation to describe the regular determinantal representation of $\det_m$ that is equivariant with respect to $\mathrm{GL}(E)$ of size $2^m - 1$ mentioned in Theorem 2.14.

Observe that $(S^k E)_{reg}$ is isomorphic to $\Lambda^k E$ as a $T^{\mathrm{GL}(E)}$-module but not as an $\mathfrak{S}_m$-module. The irreducible $\mathfrak{S}_m$-modules are indexed by partitions of m, write $[\pi]$ for the $\mathfrak{S}_m$-module associated to the partition π. Then as $\mathfrak{S}_m$-modules, $(S^k E)_{reg} = \oplus_{j=0}^{\min\{k,m-k\}} [m-j, j]$, while $\Lambda^k E = [m-k, 1^k] \oplus [m-k+1, 1^{k-1}]$. In particular these spaces are not isomorphic as $N(T^{SL(E)})$-modules.

Write $\mathcal{M}_m(\mathbb{C}) = E \otimes F^*$. Let $f_1, \ldots, f_m$ be a basis of F^*. Let ex_k denote exterior multiplication in E:

$$ex_k : E \longrightarrow (\Lambda^k E)^* \otimes (\Lambda^{k+1} E)$$
$$v \mapsto \{\omega \mapsto v \wedge \omega\}.$$

Proposition 2.16. *The following is a regular determinantal representation of $\det_m$ that is equivariant with respect to $\mathrm{GL}(E)$. Let $\mathbb{C}^n = \bigoplus_{j=0}^{m-1} \Lambda^j E$, so $n = 2^m - 1$ and $\mathrm{End}(\mathbb{C}^n) = \oplus_{0 \leq i,j \leq m-1} \mathrm{Hom}(\Lambda^j E, \Lambda^i E)$. Fix an identification $\Lambda^m E \simeq \Lambda^0 E$. Set*

$$\Lambda_0 = \sum_{k=1}^{m-1} \mathrm{Id}_{\Lambda^k E},$$

and

$$\tilde{A} = \Lambda_0 + \sum_{k=0}^{m-1} ex_k(E) \otimes f_{k+1}. \tag{13}$$

Then $\det_m = \det_n \circ \tilde{A}$ if $m \equiv 1, 2 \bmod 4$ and $\det_m = -\det_n \circ \tilde{A}$ if $m \equiv 0, 3 \bmod 4$. In bases respecting the direct sum, the linear part, other than the last term which lies in the upper right block, lies just below the diagonal blocks, and all blocks other than the upper right, the diagonal and sub-diagonal are zero.

Note the similarity with the expression (10). This will be useful for proving the results about the determinantal representations of the permanent.

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When $m = 2$ this is

$$\begin{pmatrix} 0 & -y_2^2 & y_1^2 \\ y_1^1 & 1 & 0 \\ y_2^1 & 0 & 1 \end{pmatrix}$$

agreeing with our earlier calculation of a rank four quadric. Note the minus sign in front of y_2^2 because $ex(e_2)(e_1) = -e_1 \wedge e_2$.

For example, ordering the bases of $\Lambda^2\mathbb{C}^3$ by $e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3$, the matrix for $\det_3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & y_3^3 & -y_2^3 & y_1^3 \\ y_1^1 & 1 & & & & & \\ y_2^1 & & 1 & & & & \\ y_3^1 & & & 1 & & & \\ & -y_2^2 & y_1^2 & 0 & 1 & & \\ & -y_3^2 & 0 & y_1^2 & & 1 & \\ & 0 & -y_3^2 & y_2^2 & & & 1 \end{pmatrix}.$$

We now give a regular equivariant determinantal representation of $\det_m$. Let EX_k denote the exterior multiplication

$$EX_k : E\otimes F^* \longrightarrow (\Lambda^k E\otimes\Lambda^k F^*)^*\otimes(\Lambda^{k+1}E\otimes\Lambda^{k+1}F^*)$$
$$e\otimes f \mapsto \{\omega\otimes\eta \mapsto e\wedge\omega\otimes f\wedge\eta\}.$$

Proposition 2.17. *The following is an equivariant regular determinantal representation of* $\det_m$. *Let* $\mathbb{C}^n = \bigoplus_{j=0}^{m-1}\Lambda^j E\otimes\Lambda^j F^*$, *so* $n = \binom{2m}{m} - 1 \sim 4^m$ *and* $\mathrm{End}(\mathbb{C}^n) = \oplus_{0\le i,j\le m}\mathrm{Hom}(\Lambda^j E\otimes\Lambda^j F^*, \Lambda^i E\otimes\Lambda^i F^*)$. *Fix an identification* $\Lambda^m E\otimes\Lambda^m F^* \simeq \Lambda^0 E\otimes\Lambda^0 F^*$. *Set*

$$\Lambda_0 = \sum_{k=1}^{m-1}\mathrm{Id}_{\Lambda^k E\otimes\Lambda^k F^*}$$

and define

$$\tilde{A} = (m!)^{\frac{-1}{n-m}}\Lambda_0 + \sum_{k=0}^{m-1} EX_k(E\otimes F). \tag{14}$$

Then $(-1)^{m+1}\det_m = \det_n \circ \tilde{A}$.

Comparing Theorems 2.1 and 2.13, Theorems 2.8 and 2.14, Propositions 2.9 and 2.16 and Propositions 2.10 and 2.17, one can see that $\det_m$ and perm_m have the same behavior relatively to equivariant regular determinantal representations. This prompts the question: What is the regular determinantal complexity of the determinant? In particular:

Question 2.18. Let $\mathrm{rdc}(\det_m)$ be the smallest value of n such that there exist affine linear maps $\tilde{A}_m : \mathbb{C}^{m^2} \to \mathbb{C}^{n^2}$ such that

$$\det_m = \det_n \circ \tilde{A}_m \text{ and } \mathrm{rank}\tilde{A}(0) = n - 1. \tag{15}$$

What is the growth of $\mathrm{rdc}(\det_m)$?[2]

[2] After this paper was posted on arXiv, this question was answered in [13]: $\mathrm{rdc}(\det_m) = O(m^3)$.

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2.6. Overview

In §3 we establish basic facts about determinantal representations and review results about algebraic groups. The proofs of the results are then presented in increasing order of difficulty, beginning with the easy case of quadrics in §4, then the case of the determinant in §5, and concluding with the permanent in §6.

3. Preliminaries on symmetries

Throughout this section $P \in S^m V^*$ is a polynomial and

$$\tilde{A} = \Lambda + A : V \longrightarrow \mathcal{M}_n(\mathbb{C})$$

is a determinantal representation of P.

3.1. Notation

For an affine algebraic group G, G° denotes the connected component of the identity. The homomorphisms from G to $\mathbb{C}^*$ are called *characters* of G. They form an abelian group denoted by $X(G)$. The law in $X(G)$ is denoted additively (even if it comes from multiplication in $\mathbb{C}^*$). If $G \simeq (\mathbb{C}^*)^{\times r}$ is a torus then $X(G) \simeq \mathbb{Z}^r$.

For a vector space V, $\mathbb{P}V$ is the associated projective space. For a polynomial P on V, $\{P = 0\} = \{x \in V : P(x) = 0\}$ denotes its zero set, which is an affine algebraic variety. For $v \in V$, the differential of P at v is denoted $d_v P \in V^*$ and $\{P = 0\}_{sing} = \{x \in V : P(x) = 0 \text{ and } d_x P = 0\}$ denotes the singular locus of $\{P = 0\}$. Note that we do not consider the reduced algebraic variety, in particular if P is a square $\{P = 0\}_{sing} = \{P = 0\}$.

3.2. Representations of $\mathbb{G}_A$

The following observation plays a key role:

Lemma 3.1. *$\bar{\rho}_A(\mathbb{G}_A) \subset \mathbb{G}_P$. Moreover, for any $g \in \mathbb{G}_A$, $\chi_{\det_n}(g) = \chi_P(\bar{\rho}_A(g))$.*

Proof. Let $g \in \mathbb{G}_A$ and $v \in V$. Then

$$\begin{aligned}
(\bar{\rho}_A(g)P)(v) &= P(\bar{\rho}_A(g)^{-1}v) \\
&= \det_n(\Lambda + A(A^{-1}g^{-1}A(v))) \\
&= \det_n(g^{-1}(\Lambda + A(v)) \\
&= (g \det_n)(\tilde{A}(v)) \\
&= \chi_{\det_n}(g) \det_n(\tilde{A}(v)) \\
&= \chi_{\det_n}(g) P(v).
\end{aligned}$$

The lemma follows. □

3.3. Normal form for Λ

Lemma 3.2. [29] *Let $P \in S^m V^*$ be a polynomial. If codim$(\{P = 0\}_{sing}, V) \geq 5$, then any determinantal representation $\tilde{A}$ of P is regular.*

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Proof. Consider the affine variety $\{\det_n = 0\}$. The singular locus of $\{\det_n = 0\}$ is the set of matrices of rank at most $n-2$, and hence has codimension 4 in $\mathcal{M}_n(\mathbb{C})$.

For any $v \in V$, we have $d_v P = d_v(\det_n \circ \tilde{A}) = d_{\tilde{A}(v)}(\det_n) \circ A$. In particular, if $\tilde{A}(v) \in \{\det_n = 0\}_{sing}$ then $v \in \{P = 0\}_{sing}$.

But the set of v such that $\mathrm{rk}(\tilde{A}(v)) \le n-2$ is either empty, or its codimension is at most 4. The assumption of the lemma implies that $\mathrm{rk}(\tilde{A}(v)) \ge n-1$ for any $v \in V$. In particular $\mathrm{rk}(\tilde{A}(0)) = n-1$. □

In [29], von zur Gathen showed that $\mathrm{codim}(\{\mathrm{perm}_m = 0\}_{sing}, \mathbb{C}^{m^2}) \ge 5$.

Let $\Lambda_{n-1} \in \mathcal{M}_n(\mathbb{C})$ be the matrix with 1 in the $n-1$ last diagonal entries and 0 elsewhere. Any determinantal representation $\tilde{A}$ of P of size n with $\mathrm{rank}(\tilde{A}(0)) = n-1$ can be transformed (by multiplying on the left and right by constant invertible matrices) to a determinantal representation of P satisfying $\tilde{A}(0) = \Lambda_{n-1}$.

3.4. An auxiliary symmetry group

Recall the group $\mathbb{G}_{\det_n} \subset \mathrm{GL}_{n^2}$ from Equation (4). The following group plays a central role in the study of regular equivariant determinantal representations:

$$\mathbb{G}_{\det_n,\Lambda_{n-1}} = \{g \in \mathbb{G}_{\det_n} \mid g \cdot \Lambda_{n-1} = \Lambda_{n-1}\}.$$

Let $\mathbb{H} \subset \mathbb{C}^n$ denote the image of Λ_{n-1} and $\ell_1 \in \mathbb{C}^n$ its kernel. Write ℓ_2 for ℓ_1 in the target $\mathbb{C}^n$. Then $\mathcal{M}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$. Let $\mathrm{transp} \in \mathrm{GL}(\mathcal{M}_n(\mathbb{C}))$ denote the transpose.

Lemma 3.3. *The group $\mathbb{G}_{\det_n,\Lambda_{n-1}}$ is*

$$\{M \mapsto \begin{pmatrix} \lambda_2 & 0 \\ v_2 & g \end{pmatrix} M \begin{pmatrix} 1 & \phi_1 \\ 0 & g \end{pmatrix}^{-1} \mid g \in \mathrm{GL}(\mathbb{H}), v_2 \in \mathbb{H}, \phi_1 \in \mathbb{H}^*, \lambda_2 \in \mathbb{C}^*\} \cdot \langle \mathrm{transp} \rangle.$$

In particular, it is isomorphic to

$$[\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H}) \ltimes (\mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2)] \rtimes \mathbb{Z}_2.$$

Proof. First note that $\mathrm{transp}(\Lambda_{n-1}) = \Lambda_{n-1}$, so it is sufficient to determine the stabilizer of Λ_{n-1} in $\mathbb{G}^{\circ}_{\det_n}$. Let $A, B \in \mathrm{GL}_n(\mathbb{C})$ such that $A\Lambda_{n-1}B^{-1} = \Lambda_{n-1}$. Since A stabilizes the image of Λ_{n-1} and B stabilizes the Kernel of Λ_{n-1} we have

$$A = \begin{pmatrix} \lambda_2 & 0 \\ v_2 & g_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda_1 & \phi_1 \\ 0 & g_1 \end{pmatrix},$$

for some $\lambda_1 \in \mathrm{GL}(\ell_1)$, $\lambda_2 \in \mathrm{GL}(\ell_2)$, $g_1, g_2 \in \mathrm{GL}(\mathbb{H})$, $v_2 \in \mathbb{H}$ and $\phi_1 \in \mathbb{H}^*$. The identity $A\Lambda_{n-1}B^{-1} = \Lambda_{n-1}$ is now equivalent to $g_1 = g_2$. Multiplying A and B by $\lambda_1^{-1}\mathrm{Id}_n$ gives the result. □

3.5. Facts about complex algebraic groups

Let G be an affine complex algebraic group. The group G is

- *reductive* if every G-module may be decomposed into a direct sum of irreducible G-modules,
- *unipotent* if it is isomorphic to a subgroup of the group U_n of upper triangular matrices with 1's on the diagonal.

Given a complex algebraic group G, there exists a maximal normal unipotent subgroup $R^u(G)$, called the *unipotent radical*. The quotient $G/R^u(G)$ is reductive. Moreover there exist subgroups L in G such that $G = R^u(G)L$. In particular such L are reductive. Such a subgroup L is not unique, but any two such are conjugate in G (in fact by an element of $R^u(G)$). Such a subgroup L is called a *Levi factor of* G. A good reference is [23, Thm. 4. Chap. 6].

Malcev's theorem (see, e.g., [23, Thm. 5. Chap. 6]) states that fixing a Levi subgroup $L \subset G$ and given any reductive subgroup H of G, there exists $g \in R^u(G)$ such that $gHg^{-1} \subseteq L$.

For example, when G is a parabolic subgroup, e.g. $G = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, we have $L = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $R^u(G) = \begin{pmatrix} \mathrm{Id}_a & * \\ 0 & \mathrm{Id}_b \end{pmatrix}$.

A more important example for us is $R^u(\mathbb{G}_{\det_n,\Lambda_{n-1}}) = (\mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2)$ and a Levi subgroup is $L = (\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})) \rtimes \mathbb{Z}_2$.

3.6. Outline of the proofs of lower bounds

Let $P \in S^m V^*$ be either a quadric, a permanent or a determinant. Say a regular representation $\tilde{A}$ is equivariant with respect to some $G \subseteq \mathbb{G}_P$. We may assume that $\tilde{A}(0) = \Lambda_{n-1}$.

The first step consists in lifting G to $\mathbb{G}_A$. More precisely, in each case we construct a reductive subgroup $\tilde{G}$ of $\mathbb{G}_A$ such that $\bar{\rho}_A : \tilde{G} \longrightarrow G$ is finite and surjective. In a first reading, it is relatively harmless to assume that $\tilde{G} \simeq G$. Then, using Malcev's theorem, after possibly conjugating $\tilde{A}$, we may assume that $\tilde{G}$ is contained in $(\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})) \rtimes \mathbb{Z}_2$. Up to considering an index two subgroup of $\tilde{G}$ if necessary, we assume that $\tilde{G}$ is contained in $\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})$ (with the notation of Lemma 3.3).

Now, both $\mathcal{M}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$ and V (via $\bar{\rho}_A$) are $\tilde{G}$-modules. Moreover, A is an equivariant embedding of V in $\mathcal{M}_n(\mathbb{C})$. This turns out to be a very restrictive condition.

Write

$$\mathcal{M}_n(\mathbb{C}) = \begin{pmatrix} \ell_1^* \otimes \ell_2 & \mathbb{H}^* \otimes \ell_2 \\ \ell_1^* \otimes \mathbb{H} & \mathbb{H}^* \otimes \mathbb{H} \end{pmatrix}, \quad \Lambda_{n-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{\mathbb{H}} \end{pmatrix}.$$

If $m \geq 2$ the $\ell_1^* \otimes \ell_2$ coefficient of $\tilde{A}$ has to be zero. Then, since $P \neq 0$, the projection of $A(V)$ on $\ell_1^* \otimes \mathbb{H} \simeq \mathbb{H}$ has to be non-zero. We thus have a G-submodule $\mathbb{H}_1 \subset \mathbb{H}$ isomorphic to an irreducible submodule of V. A similar argument shows that there must be another irreducible G-submodule $\mathbb{H}_2 \subset \mathbb{H}$ such that an irreducible submodule of V appears in $\mathbb{H}_1^* \otimes \mathbb{H}_2$.

In each case, we can construct a sequence of irreducible sub-$\tilde{G}$-modules $\mathbb{H}_k$ of $\mathbb{H}$ satisfying very restrictive conditions. This allows us to get our lower bounds.

To prove the representations $\tilde{A}$ actually compute the polynomials we want, in the case $G = \mathbb{G}_P$, we first check that $\mathbb{G}_P$ is contained in the image of $\bar{\rho}_A$. Since P is characterized by its symmetries, we deduce that $\det_n \circ \tilde{A}$ is a scalar multiple of P. We then specialize to evaluating on the diagonal matrices in $\mathcal{M}_m(\mathbb{C})$ to determine this constant, proving in particular that it is non-zero.

4. Symmetric determinantal representations of quadrics

We continue the notation of §2.4, in particular $Q \in S^2\mathbb{C}^{2s}$ is a nondegenerate quadric.

Let $\tau : \mathbb{C}^* \times O(2s) \longrightarrow GL(2s)$ be given by $(\lambda, M) \mapsto \lambda M$. The image of τ is the group $\mathbb{G}_Q$ and $\chi_Q \circ \tau(\lambda, M) = \lambda^{-2}$.

In the expression (12) we have $\mathbb{G}_A = (\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})) \rtimes \mathbb{Z}/2\mathbb{Z}$. Since $\rho_A(\mathbb{G}_A^\circ)$ is a proper subgroup of $\mathbb{G}_Q$, this determinantal representation is not equivariant.

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More generally, let $\tilde{B}$ be any determinantal representation of Q of size $s+1$. By Lemma 3.2 rank$(\Lambda) = s$. Then the dimension of $\mathbb{G}_B$ cannot exceed that of $\mathbb{G}_{\det_{s+1},\Lambda_s}$, which is $(s+1)^2$ by Lemma 3.3. But the dimension of $\mathbb{G}_Q$ is $2s^2 - s + 1$. Hence the representation cannot respect the symmetries. (This argument has to be refined when $s = 3$, observing that the unipotent radical of $\mathbb{G}_{\det_{s+1},\Lambda_s}$ is contained in the kernel of ρ_A.)

Nonetheless, in the case of quadrics, the smallest presentation A is equivariant with respect to about "half" the symmetry, as was the case in Example 2.5. We will see this again with perm_3 but so far have no explanation.

Proof of Proposition 2.11. Let $\tilde{A} = \Lambda + A$ be an equivariant determinantal representation of Q. By Lemma 3.2, one may assume that $\tilde{A}(0) = \Lambda_{n-1}$.

We now construct an analog L of the group $\tilde{G}$ mentioned in §3.6. Start with $H = \bar{\rho}_A{}^{-1}(\mathbb{G}_Q)$. Consider a Levi decomposition $H = R^u(H)L$. Then $\bar{\rho}_A(R^u(H))$ is a normal unipotent subgroup of $\mathbb{G}_Q$. Since $\mathbb{G}_Q$ is reductive this implies that $R^u(H)$ is contained in the kernel of $\bar{\rho}_A$. In particular, $\bar{\rho}_A(L) = \mathbb{G}_Q$. Since $\mathbb{G}_Q$ is connected, $\bar{\rho}_A(L^\circ) = \mathbb{G}_Q$.

By construction L is a reductive subgroup of $\mathbb{G}_{\det_n,\Lambda_{n-1}}$. By Malcev's theorem, possibly after conjugating $\tilde{A}$, we may and will assume that L is contained in $(\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})) \rtimes \mathbb{Z}_2$. In particular L° is contained in $\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})$ and $\overline{\rho}_A(L^\circ) = \mathbb{G}_Q$.

Observe that $A(V)$ is an irreducible L°-submodule of $\mathcal{M}_n(\mathbb{C})$ isomorphic to V. Moreover, the action of L° respects the decomposition

$$\mathcal{M}_n(\mathbb{C}) = \ell_1^* \otimes \ell_2 \oplus \ell_1^* \otimes \mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2 \oplus \mathbb{H}^* \otimes \mathbb{H}.$$

The projection of $A(V)$ on $\ell_1^* \otimes \ell_2$ has to be zero, since it is L°-equivariant. Hence

$$A(V) \subset \ell_1^* \otimes \mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2 \oplus \mathbb{H}^* \otimes \mathbb{H}.$$

In matrices:

$$A(V) \subset \begin{pmatrix} 0 & \mathbb{H}^* \otimes \ell_2 \\ \ell_1^* \otimes \mathbb{H} & \mathbb{H}^* \otimes \mathbb{H} \end{pmatrix}.$$

Thus for the determinant to be non-zero, we need the projection to $\ell_1^* \otimes \mathbb{H}$ to be non-zero. Thus it must contain at least one copy of V. In particular dim $(\mathbb{H}) \geq$ dim (V); the desired inequality. $\square$

5. Proofs of the determinantal representations of the determinant

Recall our notations that E, F are complex vector spaces of dimension m and we have an identification $E \otimes F^* \simeq \mathrm{Hom}(F, E)$. A linear map $u : F \to E$ induces linear maps

$$\begin{aligned} u^{\wedge k} &: \Lambda^k F \to \Lambda^k E \\ v_1 \wedge \cdots \wedge v_k &\mapsto u(v_1) \wedge \cdots \wedge u(v_k). \end{aligned} \tag{16}$$

In the case $k = m$, $u^{\wedge m}$ is called the determinant of u and we denote it $\mathcal{D}et(u) \in \Lambda^m F^* \otimes \Lambda^m E$. The map

$$\begin{aligned} E \otimes F^* = \mathrm{Hom}(F, E) &\longrightarrow \Lambda^m F^* \otimes \Lambda^m E \\ u &\longmapsto \mathcal{D}et(u) \end{aligned}$$

is polynomial, homogeneous of degree m, and equivariant for the natural actions of $\mathrm{GL}(E) \times \mathrm{GL}(F)$.

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The transpose of u is

$$u^T : E^* \longrightarrow F^*,$$
$$\varphi \longmapsto \varphi \circ u.$$

Hence $u^T \in F^* \otimes E$ is obtained from u by switching E and F^*, and $\mathcal{D}et(u^T) \in \Lambda^m E \otimes \Lambda^m F^*$. Moreover, $\mathcal{D}et(u^T) = \mathcal{D}et(u)^T$.

Proof of Proposition 2.16. Set $P = \det_n \circ \tilde{A}$. To analyze the action of $\mathrm{GL}(E)$ on $\tilde{A}$, reinterpret $\mathbb{C}^{n*} \otimes \mathbb{C}^n$ without the identification $\Lambda^0 E \simeq \Lambda^m E$ as $(\oplus_{j=0}^{m-1} \Lambda^j E)^* \otimes (\oplus_{i=1}^m \Lambda^i E)$.

For each $u \in E \otimes F^*$, associate to $\tilde{A}(u)$ a linear map $\tilde{a}(u) : \oplus_{j=0}^{m-1} \Lambda^j E \to \oplus_{i=1}^m \Lambda^i E$. Then $\mathcal{D}et(\tilde{a}(u)) \in \Lambda^n(\oplus_{j=0}^{m-1} \Lambda^j E^*) \otimes \Lambda^n(\oplus_{i=1}^m \Lambda^i E)$. This space may be canonically identified as a $GL(E)$-module with $\Lambda^0 E^* \otimes \Lambda^m E \simeq \Lambda^m E$. (The identification $\Lambda^0 E \simeq \Lambda^m E$ allows one to identify this space with $\mathbb{C}$.) Using the maps (16), we get $\mathrm{GL}(E)$-equivariant maps

$$E \otimes F^* \xrightarrow{\ \tilde{a}\ } \mathrm{Hom}(\oplus_{j=0}^{m-1} \Lambda^j E, \oplus_{i=1}^m \Lambda^i E) \xrightarrow{\ \mathcal{D}et\ } \Lambda^m E.$$

Hence for all $u \in E \otimes F^*$ and all $g \in \mathrm{GL}(E)$,

$$\begin{aligned} \mathcal{D}et(\tilde{a}(g^{-1}u)) &= (g \cdot \mathcal{D}et)(\tilde{a}(u)) \\ &= \det(g)^{-1} \mathcal{D}et(\tilde{a}(u)). \end{aligned} \tag{17}$$

Equation (17) shows that $\mathrm{GL}(E)$ is contained in the image of $\bar{\rho}_A$.

Equation (17) also proves that P is a scalar (possibly zero) multiple of the determinant. Consider $P(\mathrm{Id}_m) = \det_n(\tilde{A}(\mathrm{Id}_m))$. Perform a Laplace expansion of this large determinant: there is only one non-zero expansion term, so P is the determinant up to a sign.

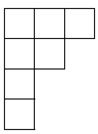
To see the sign is as asserted in Proposition 2.16, specialize to the diagonal matrices, then there is just one term. Note that y_i^i appears in the large matrix with the sign $(-1)^{i+1}$. Thus the total contribution of these signs is (-1) if $m \equiv 2, 3 \bmod 4$ and $(+1)$ if $m \equiv 0, 1 \bmod 4$.

The k-th block of Λ contributes a sign of $(-1)^{\binom{m}{k}-1}$ if we perform a left to right Laplace expansion, except for the last which contributes $(1)^{m-1}$. (The terms are negative because the $Id_{\binom{m}{k}-1}$ will begin in the left-most column each time, but, except for the last block, it begins in the second row.) Thus the total contribution of Λ to the sign is $(-1)^{\sum_{k=1}^{m-2}[\binom{m}{k}-1]} = 1$.

The slot of each $\pm y_i^i$ except y_m^m (whose slot always contributes positively in the Laplace expansion) contributes a $(-1)^{\binom{m}{i-1}}$ (it is always left-most, but appears $\binom{m}{i-1}+1$ slots below the top in the remaining matrix). Thus the total contribution from these slots is $(-1)^{\sum_{i=0}^{m-2}\binom{m}{i}} = (-1)^{2^m - m - 1} = (-1)^{m+1}$.

Thus the total sign is -1 if $m \equiv 0, 3 \bmod 4$ and $+1$ if $m \equiv 1, 2 \bmod 4$. $\square$

The irreducible polynomial representations of $\mathrm{GL}(E)$ are indexed by partitions of length at most m. Given such a partition π, we denote by $S_\pi E$ the corresponding irreducible $\mathrm{GL}(E)$-module. For example, $S_d E = S^d E$ and $S_{1^j} E = \bigwedge^j E$ (for any $1 \le j \le m$). To a partition $\pi = (p_1, \ldots, p_m)$ we associate a Young diagram, a collection of left-aligned boxes with p_j boxes in the j-th row. The number of boxes is denoted by $|\pi|$. For example, the Young diagram for $\pi = (3, 2, 1, 1) =: (3, 2, 1^2)$ is



and $|\pi| = 7$.

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A *Young tableau* is obtained by filling in the boxes of the Young diagram with integers $\{1, \ldots, m\}$. The Young diagram is called the shape of the tableau. A tableau is called *semistandard*, if the entries weakly increase along each row and strictly increase down each column. For example, for $\pi = (3, 2, 1, 1) =: (3, 2, 1^2)$ and $m = 4$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

is a semistandard Young tableau. Moreover, the dimension of $S_\pi E$ is number of semistandard Young tableaux of shape π. An easy consequence is that

$$\dim\,(S_\pi E) \geq \binom{m}{|\pi|} \tag{18}$$

Given a partition $\pi = (p_1, \ldots, p_m)$, the Pieri formula asserts that

$$E \otimes S_\pi E = \oplus_\mu S_\mu E, \tag{19}$$

where μ ranges over all partitions with at most m parts obtained from π by adding one box.

Any polynomial simple $\mathrm{GL}(E)$-module is irreducible for the action of $\mathrm{SL}(E)$. Moreover, any simple $\mathrm{SL}(E)$-module can be obtained in such a way. Finally, $S_\pi E$ and $S_\lambda E$ are isomorphic as $\mathrm{SL}(E)$- modules if and only if $\lambda \in \pi + \mathbb{Z}1^m$. For example, $S_{1^m}E = \bigwedge^m E$ is trivial as an $\mathrm{SL}(E)$-module and $S_{1^{m-1}}E = \bigwedge^{m-1} E \simeq E^*$ as an $\mathrm{SL}(E)$-module.

Proof of Theorem 2.13. Fix two m-dimensional vector spaces E and F and consider $\det_n$ as an element of $S^m(F^* \otimes E)^*$. Let $\tilde{A} = \Lambda + A : F^* \otimes E \to \mathcal{M}_n(\mathbb{C})$ be a regular determinantal representation of $\det_m$ that is equivariant with respect to $\mathrm{GL}(E)$. It remains to prove that $n \geq 2^m - 1$.

We may assume $\Lambda = \Lambda_{n-1}$ and identify $\mathcal{M}_n(\mathbb{C})$ with $(\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$. As in the proof of Proposition 2.11, after possibly conjugating $\tilde{A}$, we construct a connected reductive subgroup L of $\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})$ mapping onto $\mathrm{GL}(E)$ by $\bar{\rho}_A$.

We have an action of L on $\mathcal{M}_n(\mathbb{C})$, but we would like to work with $\mathrm{GL}(E)$. Towards this end, there exists a finite cover $\tau : \tilde{L} \longrightarrow L$ that is isomorphic to the product of a torus and a product of simple simply connected groups. In particular there exists a subgroup of $\tilde{L}$ isomorphic to $\mathbb{C}^* \times \mathrm{SL}(E)$ such that $\bar{\rho}_A \circ \tau(\mathbb{C}^* \times \mathrm{SL}(E)) = \mathrm{GL}(E)$. The group $\mathbb{C}^* \times \mathrm{SL}(E)$ acts trivially on ℓ_1, on ℓ_2 by some character and on $\mathbb{H}$. It acts on $\mathcal{M}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$ accordingly.

Consider a decomposition

$$\mathbb{H} = \mathbb{H}_1 \oplus \cdots \oplus \mathbb{H}_k$$

of $\mathbb{H}$ as a sum of irreducible $\mathrm{SL}(E)$-module. Let λ_i be partitions of length at most $m-1$ such that $\mathbb{H}_i \simeq S_{\lambda_i} E$ as an $\mathrm{SL}(E)$-module. Note that

$$(\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H}) = \ell_1^* \otimes \ell_2 \oplus \bigoplus_{1 \leq i \leq k} \ell_1^* \otimes \mathbb{H}_i \oplus \bigoplus_{1 \leq i,j \leq k} \mathbb{H}_j^* \otimes \mathbb{H}_i \oplus \bigoplus_{1 \leq j \leq k} \mathbb{H}_j^* \otimes \ell_2. \tag{20}$$

According to the decomposition (20), we have a decomposition of A as $A = \sum_{0 \leq i,j \leq k} A_{ij}$ where

$$\begin{array}{ll} A_{00} \in \mathrm{Hom}(F^* \otimes E, \ell_1^* \otimes \ell_2) & A_{i0} \in \mathrm{Hom}(F^* \otimes E, \ell_1^* \otimes \mathbb{H}_i) \\ A_{0j} \in \mathrm{Hom}(F^* \otimes E, \mathbb{H}_j^* \otimes \ell_2) & A_{ij} \in \mathrm{Hom}(F^* \otimes E, \mathbb{H}_j^* \otimes \mathbb{H}_i) \end{array}$$

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for any $1 \leq i, j \leq k$. Since each A_{ij} is $\mathrm{SL}(E)$-equivariant, $A_{00} = 0$ and

- if $A_{0j} \neq 0$ then $\mathbb{H}_j \simeq E^*$;
- if $A_{i0} \neq 0$ then $\mathbb{H}_i \simeq E$;
- if $A_{ij} \neq 0$ then E is a submodule of $\mathbb{H}_j^* \otimes \mathbb{H}_i$ and $S_{\lambda_i} E \subset E \otimes S_{\lambda_j} E$. By formula (19), this implies that, modulo 1^m, λ_i is obtained from λ_j by adding one box.

The first column of $\tilde{A}$ cannot be zero, hence there exists i such that $A_{i0} \neq 0$. Up to relabeling, we assume that $A_{10} \neq 0$ and that $\mathbb{H}_1 \simeq E$. Similarly, there exists j such that $A_{0j} \neq 0$. If $m = 2$, it is possible that $j = 1$ since $E \simeq E^*$. In this case, $\det_2$ is a quadratic form, and we recover its determinantal representation of size 3.

Assume now that $m \geq 3$, in particular that E and E^* are not isomorphic as $\mathrm{SL}(E)$-modules and $j \neq 1$. Up to relabeling, we assume that $A_{0k} \neq 0$ and that $\mathbb{H}_k \simeq E^*$.

If $A_{i1} = 0$ for any i then one can expand the columns corresponding to $\mathbb{H}_1^*$, one sees that $\det_m$ is equal to the determinant in $(\ell_1 \oplus \oplus_{j\geq 2} \mathbb{H}_j)^* \otimes (\ell_2 \otimes \oplus_{j\geq 2} \mathbb{H}_j)$, and we can restart the proof suppressing $\mathbb{H}_1$. Then there exists i such that $E \subset \mathbb{H}_1^* \otimes \mathbb{H}_i$: $|\lambda_i| = 2$, since $m \geq 3$. In particular $i \neq 1$ and we may assume that $i = 2$. Continuing, we get $|\lambda_i| = i$ for any $i = 1, \dots m-1$. Then

$$n \geq 1 + \sum_{i=1}^{m-1} \dim\left(S_{\lambda_i} E\right) \geq \sum_{i=1}^{m-1} \binom{m}{i} = 2^m - 1,$$

by inequality (18). □

Proof of Proposition 2.17. Write $\tilde{A}$ as

$$\tilde{a} : E \otimes F^* \longrightarrow \left(\oplus_{i=1}^m \Lambda^i E \otimes \Lambda^i F^* \right) \otimes \left(\oplus_{j=0}^{m-1} \Lambda^j E \otimes \Lambda^j F^* \right)^*.$$

For $u \in E \otimes F^*$, $\mathcal{D}et(\tilde{a}(u))$ belongs to

$$\left(\Lambda^n \oplus_{j=0}^{m-1} \Lambda^j E \otimes \Lambda^j F^* \right)^* \otimes \left(\Lambda^n \oplus_{i=1}^m \Lambda^i E \otimes \Lambda^i F^* \right),$$

which may be canonically identified with

$$(\Lambda^0 E \otimes \Lambda^0 F^*)^* \otimes \Lambda^m E \otimes \Lambda^m F^* \simeq \Lambda^m E \otimes \Lambda^m F^*$$

These identifications determine $\mathrm{GL}(E) \times \mathrm{GL}(F)$-equivariant polynomial maps

$$E \otimes F^* \xrightarrow{\tilde{a}} \left(\oplus_{i=1}^m \Lambda^i E \otimes \Lambda^i F^* \right) \otimes \left(\oplus_{j=0}^{m-1} \Lambda^j E \otimes \Lambda^j F^* \right)^* \xrightarrow{\mathcal{D}et} \Lambda^m E \otimes \Lambda^m F^*.$$

Choose bases $\mathcal{B}_E$ and $\mathcal{B}_F$, respectively of E and F to identify $E \otimes F^*$ with $\mathcal{M}_m(\mathbb{C})$. Choose a total order on the subsets of $\mathcal{B}_E \times \mathcal{B}_F$, to get bases of each $\Lambda^j E \otimes \Lambda^j F^*$ and maps

$$\mathcal{M}_m(\mathbb{C}) \xrightarrow{\tilde{A}} \mathcal{M}_n(\mathbb{C}) \xrightarrow{\det_n} \mathbb{C}.$$

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As in the proof of Proposition 2.16, this implies that $\mathrm{GL}(E) \times \mathrm{GL}(F)$ belongs to the image of $\bar{\rho}_A$, so $P = \det_n \circ \tilde{A}$ is a scalar multiple of $\det_m$.

To see it is the correct multiple, specialize to the diagonal matrices. Re-order the rows and columns so that all non-zero entries of $A(\mathcal{M}_m(\mathbb{C}_m))$ appear in the upper-left corner. Note that since we made the same permutation to rows and columns this does not change the sign. Also note that since we have diagonal matrices, there are only plus signs for the entries of $A(\mathcal{M}_m(\mathbb{C}_m))$. In fact this upper-left corner is exactly Grenet's representation for

$$\mathrm{perm}_m \begin{pmatrix} y_1^1 & \cdots & y_m^m \\ & \vdots & \\ y_1^1 & \cdots & y_m^m \end{pmatrix}$$

which is $m!(y_1^1 \cdots y_m^m)$. Finally note that each term in an expansion contains $n-m$ elements of Λ_0 to conclude.

It remains to prove that transp belongs to the image of $\bar{\rho}_A$. The following diagram is commutative:

$$\begin{tikzcd}
E \otimes F^* \arrow[r, "\tilde{a}"] \arrow[d, "\mathrm{transp}"] & \left(\oplus_{i=1}^m \Lambda^i E \otimes \Lambda^i F^*\right) \otimes \left(\oplus_{j=0}^{m-1} \Lambda^j E \otimes \Lambda^j F^*\right)^* \arrow[r, "\mathcal{D}et"] \arrow[d, "\Delta\,\mathrm{transp}"] & \Lambda^m E \otimes \Lambda^m F^*. \arrow[d, "\mathrm{transp}"] \\
F^* \otimes E \arrow[r] & \left(\oplus_{i=1}^m \Lambda^i F^* \otimes \Lambda^i E\right) \otimes \left(\oplus_{j=0}^{m-1} \Lambda^j F^* \otimes \Lambda^j E\right)^* \arrow[r, "\mathcal{D}et"] & \Lambda^m F^* \otimes \Lambda^m E,
\end{tikzcd}$$

where Δ transp is the transposition on each summand.

Using $\mathcal{B}_E$ and $\mathcal{B}_F$, we identify the 6 spaces with matrix spaces. The first vertical map becomes the transposition from $\mathcal{M}_m(\mathbb{C})$ to itself. The last vertical map becomes the identity on $\mathbb{C}$. The middle vertical map is the endomorphism of $\mathcal{M}_n(\mathbb{C})$ corresponding to bijections between bases of spaces $\Lambda^j E \otimes \Lambda^j F^*$ and $\Lambda^j F^* \otimes \Lambda^j E$. It follows that there exist two permutation matrices $B_1, B_2 \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\forall M \in \mathcal{M}_m(\mathbb{C}) \qquad \tilde{A}(M^T) = B_1 \tilde{A}(M) {B_2}^{-1},$$

proving that transp belongs to the image of $\bar{\rho}_A$. □

6. Proofs of results on determinantal representations of perm_m

Proofs of Propositions 2.9 and 2.10. The maps $s_k(v) : (S^k E)_{reg} \to (S^{k+1} E)_{reg}$ are related to the maps $ex_k(v) : \Lambda^k E \to \Lambda^{k+1} E$ as follows. The sources of both maps have bases indexed by multi-indices $I = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le m$, and similarly for the targets. The maps are the same on these basis vectors except for with $s_k(v)$ all the coefficients are positive whereas with $ex_k(v)$ there are signs. Thus the polynomial computed by (10) is the same as the polynomial computed by (13) except all the y_j^i appear positively. Reviewing the sign calculation, we get the result.

The maps S_k and EX_k are similarly related and we conclude this case similarly. □

Remark 6.1. The above proof can be viewed more invariantly in terms of the Young–Howe duality functor described in [2].

Proof of Theorem 2.8. Write $E, F = \mathbb{C}^m$. Let $\tilde{A}$ be a determinantal representation of perm_m such that $\tilde{A}(0) = \Lambda_{n-1}$. Embed $N(T^{\mathrm{GL}(E)})$ in $\mathrm{GL}(\mathrm{Hom}(F, E))$ by $g \longmapsto \{M \mapsto gM\}$. We assume the image of $\bar{\rho}_A$ contains $N(T^{\mathrm{GL}(E)})$. Set $N = N(T^{\mathrm{GL}(E)})$ and $T = T^{\mathrm{GL}(E)}$.

As in the proof of Theorem 2.14, we get a reductive subgroup L of $(\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})) \rtimes \mathbb{Z}_2$ mapping onto N by $\bar{\rho}_A$. In the determinant case, at this point we dealt with the universal cover of the connected reductive group $\mathrm{GL}(E)$. Here the situation is more complicated for two reasons. First, there is no "finite universal cover" of $\mathfrak{S}_m$ (see e.g. [14,25]). Second, since our group is not connected, we will have to deal with the factor $\mathbb{Z}_2$ coming from transposition, which will force us to work with a subgroup of N. Fortunately this will be enough for our purposes.

We first deal with the $\mathbb{Z}_2$: Since $L/(L \cap \mathbb{G}^\circ_{\det_n, \Lambda_{n-1}})$ embeds in $\mathbb{G}_{\det_n, \Lambda_{n-1}}/\mathbb{G}^\circ_{\det_n, \Lambda_{n-1}} \simeq \mathbb{Z}_2$, the subgroup $L \cap \mathbb{G}^\circ_{\det_n, \Lambda_{n-1}}$ has index 1 or 2 in L. Since the alternating group $\mathfrak{A}_m$ is the only index 2 subgroup of $\mathfrak{S}_m$, $\bar{\rho}_A(L \cap \mathbb{G}^\circ_{\det_n, \Lambda_{n-1}})$ contains $T \rtimes \mathfrak{A}_m \subset N$. In any case, there exists a reductive subgroup L' of L such that $\bar{\rho}_A(L') = T \rtimes \mathfrak{A}_m \subset N$.

Next we deal with the lack of a lift. We will get around this by showing we may label irreducible L' modules only using labels from $\bar{\rho}_A(L') = T^{\mathrm{GL}(E)} \rtimes \mathfrak{A}_m$.

The connected reductive group L'° maps onto T. Let Z denote the center of L'°; then $\bar{\rho}_A(Z^\circ) = T$. In particular the character group $X(T)$ may be identified with a subgroup of the character group $X(Z^\circ)$. The action of L' by conjugation on itself induces an action of the finite group L'/L'° on Z°. Moreover, the morphism $\bar{\rho}_A$ induces a surjective map $\pi_A : L'/L'^\circ \longrightarrow \mathfrak{A}_m$. These actions are compatible in the sense that for $g \in L'/L'^\circ$, $t \in T$, $z \in Z^\circ$ and $\sigma \in \mathfrak{A}_m$ satisfying $\pi_A(g) = \sigma$ and $\bar{\rho}_A(z) = t$,

$$\sigma \cdot t = \bar{\rho}_A(g \cdot z).$$

In particular, both the kernel of $\bar{\rho}_A$ restricted to Z° and $X(T)$ are stable under the (L'/L'°)-action.

Let $\Gamma_{\mathbb{Q}}$ be a complement of the subspace $X(T) \otimes \mathbb{Q}$ in the vector space $X(Z^\circ) \otimes \mathbb{Q}$ stable under the action of (L'/L'°). Set $\Gamma = \Gamma_{\mathbb{Q}} \cap X(Z^\circ)$ and $\tilde{T} = \{t \in Z^\circ : \forall \chi \in \Gamma \ \chi(t) = 1\}$. Then $\tilde{T}$ is a subtorus of Z° and the restriction of $\bar{\rho}_A$ to $\tilde{T}$ is a finite morphism onto T.

The character group $X(\tilde{T})$ may be identified with $X(Z^\circ)/\Gamma$ by restriction. Then $X(T)$ may be identified with a subgroup of $X(\tilde{T})$ of finite index. Hence there exists a natural number k_0 such that $k_0 X(\tilde{T}) \subset X(T) \subset X(\tilde{T})$.

Let W be an irreducible representation of L'. It decomposes under the action of $\tilde{T}$ as

$$W = \oplus_{\chi \in X(\tilde{T})} W^\chi.$$

Set $\mathrm{Wt}(\tilde{T}, W) = \{\chi \in X(\tilde{T}) : W^\chi \neq \{0\}\}$. The group L' acts by conjugation on $\tilde{T}$ and so on $X(\tilde{T})$. By the rigidity of tori (see e.g., [12, §16.3]), L'° acts trivially. Hence, the finite group L'/L'° acts on $\tilde{T}$ and $X(\tilde{T})$. For any $t \in \tilde{T}$, $h \in L'$ and $v \in W$, $thv = h(h^{-1}th)v$. Hence $\mathrm{Wt}(\tilde{T}, W)$ is stable under the action of L'/L'°. For $\chi \in \mathrm{Wt}(\tilde{T}, W)$, the set $\oplus_{\sigma \in L'/L'^\circ} W^{\sigma \cdot \chi}$ is stable under the action L'. By irreducibility of W, one deduces that $\mathrm{Wt}(\tilde{T}, W)$ is a single (L'/L'°)-orbit. Then $k_0 \, \mathrm{Wt}(\tilde{T}, W) \subset X(T)$ is an $\mathfrak{A}_m$-orbit.

We are now in a position to argue as in §3.6.

Let ε_i denote the character of T that maps an element of T on its ith diagonal entry. The set $\{a_1\varepsilon_1 + \cdots + a_m\varepsilon_m : a_1 \geq \cdots \geq a_{m-1} \text{ and } a_{m-2} \geq a_m\}$ is a fundamental domain of the action of $\mathfrak{A}_m$ on $X(T^{\mathrm{GL}(E)})$. Such a weight is said to be $\mathfrak{A}_m$-dominant. Hence, there exists a unique $\mathfrak{A}_m$-dominant weight χ_W such that $k_0 \, \mathrm{Wt}(\tilde{T}, W) = \mathfrak{A}_m . \chi_W$.

Summary of the properties of L', $\tilde{T}$ and k_0:

(1) L' is a reductive subgroup of $\mathrm{GL}(\ell_2) \times \mathrm{GL}(\mathbb{H})$;
(2) $\bar{\rho}_A(L') = \mathfrak{A}_m \ltimes T$;

(3) $\tilde{T}$ is a central subtorus of $(L')^\circ$;
(4) $\bar{\rho}_A : \tilde{T} \longrightarrow T$ is finite and surjective, inducing embeddings $k_0 X(\tilde{T}) \subset X(T) \subset X(\tilde{T})$;
(5) For any irreducible representation W of L' there exists a unique $\mathfrak{A}_m$-dominant weight χ_W such that $k_0 \operatorname{Wt}(\tilde{T}, W) = \mathfrak{A}_m.\chi_W$.
(6) For the standard representation E of L' through $\bar{\rho}_A$, $\chi_E = k_0\varepsilon_1$.

The action of L' on $\mathcal{M}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$ respects the decomposition

$$\mathcal{M}_n(\mathbb{C}) = \ell_1^* \otimes \ell_2 \oplus \ell_1^* \otimes \mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2 \oplus \mathbb{H}^* \otimes \mathbb{H}.$$

The image of A is an L'-module isomorphic to the sum of m copies of E. In particular, its projection on $\ell_1^* \otimes \ell_2$ has to be zero. Hence

$$A(V) \subset \ell_1^* \otimes \mathbb{H} \oplus \mathbb{H}^* \otimes \ell_2 \oplus \mathbb{H}^* \otimes \mathbb{H}.$$

As was the case before, for the determinant to be non-zero, we need the projection to $\ell_1^* \otimes \mathbb{H}$ to be non-zero, so it must contain at least one copy $\mathbb{H}_1$ of E.

Assume first that $\mathbb{H}_1^* \otimes \ell_2 \simeq E$. This happens only if $m = 2$, where perm_m is a quadric.

Assume now that $\mathbb{H}_1^* \otimes \ell_2 \not\simeq E$. Choose an L'-stable complement $\mathbb{S}_1$ of $\mathbb{H}_1$ in $\mathbb{H}$. If the projection of $A(V)$ on $\mathbb{H}_1^* \otimes \mathbb{S}_1$ is zero, one can discard $\mathbb{H}_1$ and start over as in the proof of the determinant case, so we assume it contributes non-trivially. Continuing so on, one gets a sequence $\mathbb{H}_1, \ldots, \mathbb{H}_k$ of irreducible L'-submodules of $\mathbb{H}$ in direct sum such that

(1) $k \geq 2$;
(2) $E \subset \mathbb{H}_i^* \otimes \mathbb{H}_{i+1}$ for any $i = 1, \ldots, k-1$;
(3) $E \simeq \mathbb{H}_k^* \otimes \ell_2$.

Let $\gamma : \mathbb{C}^* \longrightarrow \tilde{T}$ be a group homomorphism such that $\bar{\rho}_A \circ \gamma(t) = t^{k_0} \operatorname{Id}_E$. Then $\gamma(\mathbb{C}^*)$ acts trivially on $\mathbb{H}_1^* \otimes \mathbb{H}_1$ and with weight k_0 on E. Hence the projection of $A(V)$ on $\mathbb{H}_1^* \otimes \mathbb{H}_1$ is zero. (Recall that via Λ, $\operatorname{Id}_{\mathbb{H}_1} \in \mathbb{H}_1^* \otimes \mathbb{H}_1$.) More generally the action of γ shows that the non-zero blocks of $A(V)$ are $\ell_1^* \otimes \mathbb{H}_1$, $\mathbb{H}_i^* \otimes \mathbb{H}_{i+1}$, and $\mathbb{H}_k^* \otimes \ell_2$. Consider the following picture:

$$\begin{pmatrix} 0 & \mathbb{H}_1^* \otimes \ell_2 & ... \\ E & \operatorname{Id}_{\mathbb{H}_1} & ... \\ \vdots & \mathbb{H}_1^* \otimes S_1 & ... \end{pmatrix}.$$

Write

$$k_0 \operatorname{Wt}(\tilde{T}, \mathbb{H}_i) = \mathfrak{A}_m \cdot \chi_{\mathbb{H}_i}.$$

Then

$$k_0 \operatorname{Wt}(\tilde{T}, \mathbb{H}_i^* \otimes \mathbb{H}_{i+1}) = \{-\sigma_1 \chi_{\mathbb{H}_i} + \sigma_2 \chi_{\mathbb{H}_{i+1}} : \sigma_1, \sigma_2 \in \mathfrak{A}_m\}.$$

This set has to contain $k_0 \operatorname{Wt}(\tilde{T}, E) = \{k_0 \varepsilon_i \mid i \in [m]\}$. We deduce that $\chi_{\mathbb{H}_{i+1}} = \sigma \chi_{\mathbb{H}_i} + k_0 \varepsilon_u$ for some $\sigma \in \mathfrak{A}_m$ and $u \in [m]$.

We define the length $\ell(\chi)$ of $\chi \in X(T)$ as its number of non-zero coordinates in the basis $(\varepsilon_1, \ldots, \varepsilon_m)$. Then

$$\ell(\chi_{\mathbb{H}_{i+1}}) \leq \ell(\chi_{\mathbb{H}_i}) + 1. \tag{21}$$

Observe that χ_{ℓ_2} is invariant under $\mathfrak{A}_m$. Hence $\chi_{\ell_2} = \alpha(\varepsilon_1 + \cdots + \varepsilon_m)$ for some $\alpha \in \mathbb{Z}$. The action of γ shows that $\alpha = k - 1$. We deduce that $\ell(\chi_{\mathbb{H}_k}) \geq m - 1$. Then, by inequality (21), there exists a subset $\mathbb{H}_{i_1}, \dots, \mathbb{H}_{i_{m-1}}$ of the $\mathbb{H}_j$'s with $\ell(\chi_{\mathbb{H}_{i_s}}) = s$.

We claim that $\dim(\mathbb{H}_{i_s}) \geq \binom{m}{s}$. First, $\dim(\mathbb{H}_{i_s})$ is greater or equal to the cardinality of $\mathfrak{A}_m \chi_{\mathbb{H}_{i_s}}$. Since $m \geq 3$, $\mathfrak{A}_m$ acts transitively on the subsets of $[m]$ with s elements. The claim follows.

Summing these inequalities on the dimension of the $\mathbb{H}_{i_s}$, we get

$$\dim \mathbb{H} \geq \sum_{j=0}^{m-1} \dim \mathbb{H}_{i_j} \geq \sum_{j=1}^{m-1} \binom{m}{j} = 2^m - 2. \qquad \square$$

Proof of Theorem 2.1. This proof is omitted since it is very similar to the proof of Theorem 2.13. $\square$

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