Interpolation

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there is a unique polynomial $f \in \mathbb{C}[z]$ of degree at most d such that

$$f(z_i) = a_i, \quad i = 1, \dots, d+1.$$

More generally, given any

$$z_1, \ldots, z_k \in \mathbb{C},$$

any integer multiplicities

$$m_1, \ldots, m_k \in \mathbb{N} \quad \text{with} \quad \sum m_i = d + 1,$$

and any values

$$a_{i,j} \in \mathbb{C}, \quad 1 \leq i \leq k; \quad 0 \leq j \leq m_i - 1$$

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there is a unique $f \in \mathbb{C}[z]$ of degree at most d such that

$$f^{(j)}(z_i) = a_{i,j} \quad \forall i, j.$$

Problem:

What can we say along the same lines for polynomials in several variables?

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In other words, if the space of polynomials of degree d in r variables has dimension N, can we find a polynomial with assigned values at N points $z_\alpha \in \mathbb{C}^r$?

The first thing to observe is that this problem doesn't have a uniform answer: for example, if we consider linear polynomials $ax + by + c$ in two variables, we can find one with assigned values at three points *unless* the points lie on a line.

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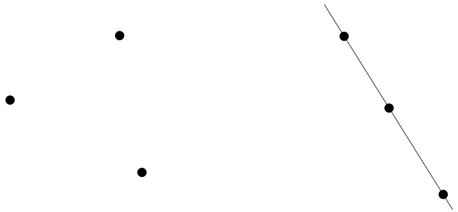

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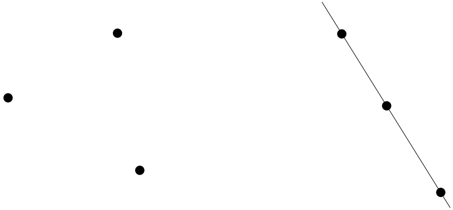

In general, linear algebra describes the answer to our problem in case $d = 1$; we want to know what we can say for $d > 1$.

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The "starting point" statement says that in case $r = 1$, for any subset $\Gamma = \{z_1, \dots, z_{d+1}\} \subset \mathbb{C}$ the evaluation map

$$\rho_\Gamma : V_d \to \mathbb{C}^{d+1} = \oplus \mathbb{C}_{z_i}$$

given by evaluation at $z_1, \dots, z_{d+1}$ is an isomorphism.

More generally, for any $n \in \mathbb{N}$ and any $\Gamma = \{z_1, \ldots, z_e\} \in \mathbb{C}$, the evaluation map

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is injective if $d + 1 \leq n$ and surjective when $d + 1 \geq n$—in other words, it has *maximal rank*.

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The same is true if we evaluate derivatives as well as values, as long as we consider all derivatives up to a certain order at each point z_i.

We can ask the analogous question for polynomials in several variables: if

$$\Gamma = \{z_\alpha \in \mathbb{C}^r\}$$

is a collection of n points, V_d the space of polynomials of degree at most d in r variables, and

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the map given by evaluation at the points of Γ, we ask:

When does ρ_Γ fail to have maximal rank, and by how much can it fail to have maximal rank?

More generally: if Γ is a configuration of points $z_\alpha \in \mathbb{C}^r$ with multiplicities m_α,

$$n = \sum_\alpha \binom{m_\alpha + r - 1}{r},$$

and

$$\rho_\Gamma : V_d \to \mathbb{C}^n$$

the map given by evaluating all derivatives up to order $m_\alpha - 1$ at z_α, again: *when does ρ_Γ fail to have maximal rank, and by how much?*

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We also denote the rank of the evaluation map ρ_Γ by $h_\Gamma(d)$; this is called the *Hilbert function* of the configuration Γ.

Our goals:

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2. give bounds on by how much they may fail: that is, how small the rank $h_\Gamma(d)$ of ρ_Γ may be.

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B) when Γ is a union of “fat points”—that is, multiplicities may be arbitrary.

As we’ll see, these two cases give rise to very different questions and answers, but there is a common thread to both.

2. Simple points

In this case, the first observation is that *general points always impose maximal conditions*—in other words, in the space $(\mathbb{C}^r)^n$ of configurations Γ of n points, those that impose maximal conditions form a dense open subset.

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In fact, if we choose a basis for the space of polynomials and write out the matrix representative for ρ_Γ, the minors of this matrix are polynomials on $\mathbb{C}^{nr}$. Thus to prove the above, we have only to show these minors are not all 0; that is, we have to exhibit a single configuration Γ that imposes maximal conditions.

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To do this, pick the points $z_i \in \Gamma$ one at a time; as long as z_{i+1} doesn't lie in the common zero locus of the polynomials of degree d vanishing at $z_1, \ldots, z_i$, Γ will impose independent conditions.

So, we ask when special configurations of points may fail to impose maximal conditions, and by how much—that is, how small $h_\Gamma(d)$ can be.

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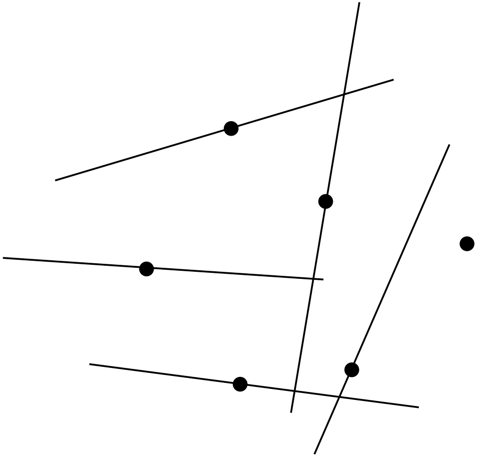
Elementary result: Any $d + 1$ distinct points in $\mathbb{C}^r$ impose independent conditions on polynomials of degree d; and $d + 2$ distinct points will fail to impose independent conditions if and only if they lie on a line.

To see this, observe that for any $p_1, \ldots, p_{d+1} \in \mathbb{C}^r$ we can find a polynomial vanishing at all but any one of the p_i by taking a product of d linear forms, each vanishing at exactly one of the points.



For the second part, observe that this will work for $d + 2$ points as long as the configuration contains three non-colinear points.

More generally, the question as posed isn't very challenging: $h_\Gamma(d)$ is minimal for Γ contained in a line.

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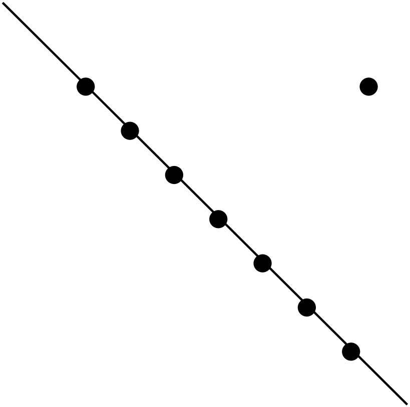
And even if we require that Γ spans, the answer isn't all that interesting: configurations Γ with $h_\Gamma(d)$ minimal will consist of $n - r + 1$ points on a line, plus $r - 1$ points off it so as to span.



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“Linear general position” here means that, for $s < r$, no affine s-plane in $\mathbb{C}^r$ contains more than $s + 1$ of the points of Γ.

We have then:

Theorem (Castelnuovo)

If $\Gamma \subset \mathbb{C}^r$ is a collection of n points in linear general position, then

$$h_\Gamma(d) \geq \min\{rd + 1, n\}.$$

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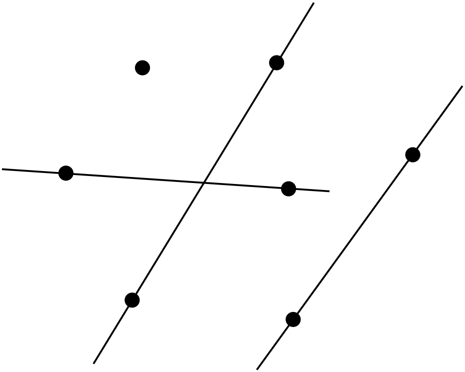
The proof is surprisingly simple: we exhibit polynomials of degree d vanishing at rd points of Γ and no others by taking products of linear polynomials each vanishing on r points.



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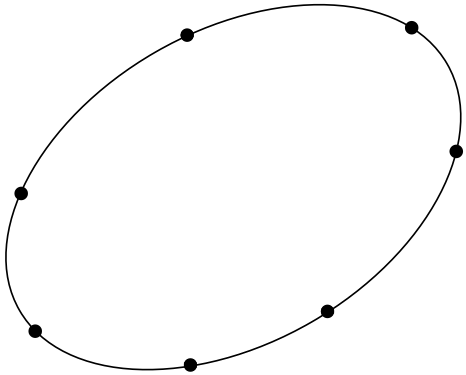
For example, in case $r = 2$ consider any configuration $\Gamma \subset C \subset \mathbb{C}^2$ lying on a conic curve C



A conic curve can be given parametrically as the image of a map

$$\begin{aligned}\phi : \mathbb{C} &\to \mathbb{C}^2\\ t &\mapsto \big(q_1(t), q_2(t)\big)\end{aligned}$$

with q_i rational functions of degree 2.

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In particular, if f vanishes on $2d + 1$ points of Γ, it must vanish identically on C and hence on all of Γ.

Thus, $h_\Gamma(d) = \min(2d + 1, n)$.

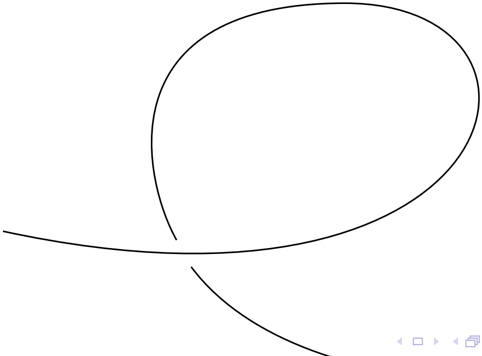
This generalizes readily to higher dimensions. First, a definition: if $f_0, \ldots, f_r$ is any basis for the vector space of polynomials of degree at most r in one variable t, we call the arc

$$t \mapsto (\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0})$$

a *rational normal curve*. For example, the arc

$$t \mapsto (t, t^2, t^3, \ldots, t^r)$$

is a rational normal curve.



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Thus, any configuration of points on a rational normal curve imposes the minimal number of conditions of polynomials of degree d.

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If $\Gamma \subset \mathbb{C}^r$ *is a collection of* $n \geq 2r + 3$ *points in linear general position, and*

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Thus we have a complete characterization of at least the extremal examples of failure to impose independent conditions.

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The second is the requirement that the number of points is $n \geq 2r + 3$. We need some lower bound on n; if n were $2r + 1$ or less the condition $h_\Gamma(2) \leq 2r + 1$ would be vacuous. But it's worth noting that the statement is actually false in case $n = 2r + 2$.

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The question is, can we extend it?

To understand where Castelnuovo's theorem is coming from, note that to any algebraic variety $X \subset \mathbb{C}^r$ we can associate a *Hilbert function* $h_X(d)$; this is defined to be the codimension, in the space of polynomials of degree d on $\mathbb{C}^r$, of the subspace of polynomials vanishing on X.

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where c is the *degree* of the curve (the number of points of intersection of X with a general hyperplane), and g its genus.

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It's from this that Castelnuovo's theorem stems: basically, it's saying that configurations with minimal Hilbert function lie on curves with minimal Hilbert function.

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We conjecture that this characterizes not just extremal configurations, but more generally ones with relatively small h_Γ.

Precisely, we have the

Conjecture

For $\alpha = 1, 2, \ldots, r - 1$*, if* $\Gamma \subset \mathbb{C}^r$ *is a collection of* $n \geq 2r + 2\alpha + 1$ *points in uniform position, and*

$$h_\Gamma(2) \leq 2r + \alpha,$$

then Γ *is contained in a curve* $C \subset \mathbb{C}^r$ *of degree at most* $r - 1 + \alpha$*.*

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"Uniform position" is a stronger form of linear general position: it means that if $\Gamma', \Gamma'' \subset \Gamma$ are subsets of the same cardinality, then $h_{\Gamma'}(d) = h_{\Gamma''}(d)\ \forall d$ (this condition for $d = 1$ is tantamount to linear general position).

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First, it's been around a while, at least in cases: Castelnuovo's theorem (the case $\alpha = 1$ of the conjecture) is from the late 19th century, and the next case $\alpha = 2$ was first established by Fano shortly after (though the conjecture wasn't formulated until around 30 years ago).

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The next case, $\alpha = 3$, was solved around 7 years ago by Ivan Petrakiev, and that's where things stand now.

We know how to classify irreducible, nondegenerate subvarieties $X \subset \mathbb{C}^r$ with $h_X(2) = 2r + \alpha$ for $\alpha \leq r - 1$. They are in fact curves of degree $r + \alpha - 1$.

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Moreover, if a configuration Γ of $n \geq 2r + 2\alpha + 1$ points does lie on such a curve C, then C will be the zero locus of the quadratic polynomials vanishing on Γ.

Thus the crux of proving the conjecture is showing that the common zero locus of the quadratic polynomials vanishing on Γ is positive-dimensional.

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Castelnuovo's interest lay in solving a classical problem: for which triples (r, n, g) does there exist a nondegenerate curve $C \subset \mathbb{C}^r$ of degree n and genus g?

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His idea was, given a curve C in r-space, to look at the intersection Γ of C with a general hyperplane: the genus of C can be read off its Hilbert function, which is in turn related to the Hilbert function of Γ. Explicitly, what we find is that

$$g(C) \;\leq\; \sum_{d=1}^{\infty} \big(n - h_\Gamma(d)\big).$$

Thus a curve of high genus must have hyperplane sections of small Hilbert function; and Castelnuovo used his Theorem to give a (sharp) upper bound $g \leq \pi(n, r)$ on the genus of a curve of degree n in r-space.

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Castelnuovo then used his “converse” to characterize curves $C \subset \mathbb{C}^r$ achieving his maximal genus: explicitly, he showed that, just as the hyperplane sections of C had to lie on a rational normal curve in $\mathbb{C}^{r-1}$, so the curve C itself had to lie on a surface $S \subset \mathbb{C}^r$ of minimal degree $r - 1$.

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We know how to describe all such surfaces, and hence how to describe curves of maximal genus.

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Thus, a proof of the conjecture would potentially yield a complete answer to the classical problem of finding the possible genera of curves of degree n in r-space.

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Configurations $\Gamma \subset \mathbb{P}^r$ of points having small Hilbert function do so because they lie on small subvarieties $X \subset \mathbb{P}^r$—meaning, subvarieties with small Hilbert function. In this case, for small d the hypersurfaces of degree d containing Γ will just be the hypersurfaces containing X; in particular, X will be the intersection of the quadrics containing Γ.

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Usually, to prove results along these lines it's enough to show the the common zero locus of the quadratic polynomials vanishing on Γ is positive-dimensional.

3. Fat points

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Recall our question: we let Γ be a configuration of k points $z_1, \ldots, z_k \in \mathbb{C}^r$ with multiplicities $m_1, \ldots, m_k \in \mathbb{N}$. We set

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Again we ask: *when does ρ fail to have maximal rank, and by how much?*

This may seem like a variant of the problem we've been considering, but there's one striking difference with the simple point case: it's *not* always the case that a general configuration Γ imposes maximal conditions on hypersurfaces of degree d!

In the simplest example of this, we ask: does there exist a quadratic polynomial in two variables with assigned values and derivatives at two points $p, q \in \mathbb{C}^2$? In other words, is the map

$$\begin{aligned} \rho : V_2 &\to \mathbb{C}^6 \\ f(x,y) &\mapsto \left(f(p), \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), f(q), \frac{\partial f}{\partial x}(q), \frac{\partial f}{\partial y}(q) \right) \end{aligned}$$

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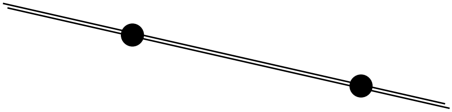
surjective?

Since both spaces are 6-dimensional, we might expect so.

But in fact ρ has a kernel: the square of the linear polynomial vanishing at p and q!



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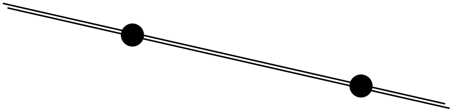

So interpolation fails in this case.

The first question is thus:

For what values of the integers r, k, $m_1, \ldots, m_k$ and d does a general configuration Γ impose maximal conditions?

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This is unknown, even for polynomials in two variables!

We do have an answer, though, in case all multiplicities are 2:

Theorem (Alexander, Hirschowitz)

A general configuration of k points with multiplicity 2 in $\mathbb{C}^r$ imposes maximal conditions on polynomials of degree d, with exactly four exceptions:

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3. $r = 3$, $k = 9$, $d = 4$
4. $r = 4$, $k = 7$, $d = 3$

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Conjecture (Harbourne-Hirschowitz)

Let $z_1, \dots, z_k \in \mathbb{C}^2$ be general, and $m_1, \dots, m_k$ arbitrary multiplicities. The corresponding configuration Γ will fail to impose maximal conditions on polynomials of degree d iff there is a curve $C \subset \mathbb{C}^2$ with

$$\sum_\alpha m_\alpha \cdot \mathrm{mult}_{z_\alpha} C \; \geq \; d \cdot \deg(C) + 2.$$

Notes:

1. If true, it gives a complete answer to our question for $r = 2$: while it may not be apparent, assuming the conjecture we can recursively list all $m_1, \ldots, m_k$ and d for which there exists such a curve.

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3. This is known when $\max\{m_i\} \leq 7$ (S. Yang)

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The content of the HH conjectures may be thought of as this: that if general multiple points in $\mathbb{C}^2$ fail to impose maximal conditions, they do so because they lie on a “small” curve—in particular, a curve C such that any polynomial of degree d satisfying the conditions vanishes on C.

4. Recasting the problem.

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So let's recast the problem: let's drop all the conditions we've put on Γ at various points above, and instead make just one assumption: that the intersection of the hypersurfaces of degree d containing Γ is zero-dimensional; in other words, Γ *is a subset of a complete intersection of* r *hypersurfaces of degree* d.

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We ask: what bounds can we give on $h_\Gamma(d)$ under this hypothesis?

One further wrinkle: instead of specifying the degree e of Γ and asking for estimates on the size of $h_\Gamma(d)$, let's turn it around: let's specify $h_\Gamma(d)$, and ask for a bound on the degree of Γ.

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Thus, the question is:

Let V be an N-dimensional vector space of polynomials of degree at most d in r variables, whose common zero locus Γ is finite. How large can the degree of Γ be?

As a first example, let's try $d = 2$ and $N = r + 1$. The question is, in effect:

As a first example, let's try $d = 2$ and $N = r + 1$. The question is, in effect:

How many common zeroes can $r + 1$ (linearly independent) quadratic polynomials in $\mathbb{C}^r$ have, if they have only finitely many common zeroes?

Theorem (Lazarsfeld)

If $Q_1, \dots, Q_{r+1}$ *are linearly independent quadrics in* $\mathbb{P}^r$*, with*

$$\Gamma = Q_1 \cap \cdots \cap Q_{r+1}$$

finite and reduced, then

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(Lazarsfeld actually answers the general question in case $N = r + 1$ under the hypothesis that Γ is reduced.)

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If $p_1, \ldots, p_8 \subset \mathbb{C}^3$ comprise the zero locus of three quadrics Q_1, Q_2, Q_3, then any quadric Q vanishing at 7 of the p_i vanishes at them all. (This is Cayley-Bacharach.)

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If $p_1, \ldots, p_{16} \subset \mathbb{C}^4$ comprise the zero locus of four quadrics $Q_1, \ldots, Q_4$, then any quadric Q vanishing at 13 of the p_i vanishes at them all. (This is Enriques-Babbage.)

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If $p_1, \ldots, p_{32} \subset \mathbb{C}^5$ comprise the zero locus of five quadrics $Q_1, \ldots, Q_5$, then any quadric Q vanishing at 25 of the p_i vanishes at them all.

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If $p_1, \ldots, p_{32} \subset \mathbb{C}^5$ comprise the zero locus of five quadrics $Q_1, \ldots, Q_5$, then any quadric Q vanishing at 25 of the p_i vanishes at them all.

and so on.

As for the general question

How many common zeroes can N (linearly independent) polynomials of degree d in $\mathbb{C}^r$ have, if they have only finitely many common zeroes?

for general N and d, we have a conjectured answer, but no proof.

Interpolation, in all its forms can be a very frustrating problem: it's completely elementary to pose the question, and we think we know what the answer should be, both philosophically and explicitly, but it seems difficult to prove.

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Still, I hope I've convinced you that it's a problem worth thinking about.

Thank you for your time and attention.