(1) Let $X \subset \mathbb{P}V = \mathbb{P}^N$ be a projective variety, and let $y \in \mathbb{P}V$ be a point. Define the *cone over X with vertex y*,

$$J(X,y) := \overline{\bigcup_{x \in X} \langle x, y \rangle}$$

where $\langle x, y \rangle$ is the projective span of x and y (a $\mathbb{P}^1$ if $x \neq y$). Show that one does not need the Zariski closure in the definition if $y \notin X$.

(2) Assume $y = [1, 0, \cdots, 0]$. Show that $P \in \mathcal{I}_{J(X,y)}$ if and only if $\frac{\partial^j P}{(\partial x_1)^j} \in \mathcal{I}_X$ for all $0 \leq j \leq \deg(P)$.

Remark 0.1. More generally for $X, Z \subset \mathbb{P}V$, one can define $J(X,Z)$, the *join of X and Z* to be

$$J(X,Z) := \overline{\bigcup_{x \in X, z \in Z} \langle x, y \rangle}.$$

When $X = Z$, $J(X,X)$ is called the *secant variety of X* and is denoted $\sigma(X) = \sigma_2(X)$. More generally, one defines $\sigma_r(X) := J(X, \sigma_{r-1}(X))$, the variety of secant $\mathbb{P}^{r-1}$'s to X. How would you find the ideals of these varieties given the ideals of X and Z?