Clay Lecture 2:

Border ranks of tensors with symmetry

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Supported by NSF grant CCF-1814254 and Clay foundation
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The classical substitution method for tensor rank

Prop. Let $T \in A{\otimes}B{\otimes}C$. Write $T = \sum_{i=1}^{\mathbf{a}} a_i \otimes M_i$, where $M_i \in B \otimes C$. Let $\mathbf{R}(T) = r$ and $M_1 \neq 0$. Then $\exists\ \lambda_2, \ldots, \lambda_{\mathbf{a}} \in \mathbb{C}$, such that $\mathbf{R}(\tilde{T}) \leq r-1$ where

$$\tilde{T} := \sum_{j=2}^{\mathbf{a}} a_j \otimes (M_j - \lambda_j M_1) \in \langle a_2, \ldots, a_{\mathbf{a}} \rangle{\otimes}B{\otimes}C.$$

Proof: $\mathbf{R}(T) = r \Rightarrow \exists\ X_1, \ldots, X_r$ rank one and d_j^i so $M_j = \sum d_j^i X_i$. $M_1 \neq 0 \Rightarrow$ WLOG $d_1^1 \neq 0$.

Take $\lambda_j = d_j^1/d_1^1 \Rightarrow \tilde{T}(\langle a_2, \ldots, a_{\mathbf{a}} \rangle^*) \subset \langle X_2, \ldots, X_r \rangle \Rightarrow$ $\mathbf{R}(\tilde{T}) \leq r-1$. □

Alexeev-Forbes-Tsimmerman: Used to find explicit sequence of tensors with $\mathbf{R}(T^m) \geq 3m - \log(m) + 1$ (best known over $\mathbb{C}$). Used tensors with nice combinatorial structure

The substitution method rephrase

Prop. Let $T \in A{\otimes}B{\otimes}C$ be A-concise. Fix a line $[\alpha] \in \mathbb{P}A^*$. Then

$$\mathbf{R}(T) \geq \min_{A' \subset A^*, \text{hyperplane}, \alpha \not\subset A'} \mathbf{R}(T|_{A' \otimes B^* \otimes C^*}) + 1$$

Compare: above $[\alpha] = \langle a_2, \ldots, a_{\mathbf{a}} \rangle^{\perp}$

$$\tilde{T} := \sum_{j=2}^{\mathbf{a}} a_j \otimes (M_j - \lambda_j M_1) \in \langle a_2, \ldots, a_{\mathbf{a}} \rangle{\otimes}B{\otimes}C.$$

The substitution method: Border rank version

Prop. (L-Michalek, Bläser-Lysikov, 2016) Let $T \in A \otimes B \otimes C$ be A-concise.

Then

$$\underline{\mathbf{R}}(T) \geq \min_{A' \subset A^*, \text{hyperplane}} \underline{\mathbf{R}}(T|_{A' \otimes B^* \otimes C^*}) + 1$$

Proof: Say $\underline{\mathbf{R}}(T) = r$ so $T = \lim_{t \to 0} T_t$.

By rank version, $\exists\ a(t) \in A$, such that $\mathbf{R}(T_t|_{a(t)^\perp \otimes B^* \otimes C^*}) \geq r - 1$
Let $[a] = \lim_{t \to 0}[a(t)]$ to conclude with $A' = a^\perp$.

How to use? Idea: if T has symmetry, can restrict search of A''s.

Weights (Generalized eigenvalues)

Fix bases and let $\mathbb{T} \subset GL(V)$ denote the diagonal matrices,

$\mathbb{T}$: maximal torus.

Write $t := \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{\mathbf{v}} \end{pmatrix}$ then $te_j = t_j e_j$. More generally

$$t(e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_{\mathbf{v}}^{\otimes p_{\mathbf{v}}}) = t_1^{p_1} \cdots t_{\mathbf{v}}^{p_{\mathbf{v}}} e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_{\mathbf{v}}^{\otimes p_{\mathbf{v}}}$$

Say *weight vector* of *weight* $(p_1, \ldots, p_{\mathbf{v}})$, :=simultaneous eigenvector $\forall t \in \mathbb{T}$. Note $\mathfrak{S}_d$ preserves weight.

Let $\mathbb{B} \subset GL(V)$ upper triangular matrices (Borel subgroup). A weight line $x \in \mathbb{P}V^{\otimes d}$ is a *highest weight line* if $\mathbb{B} \cdot x = x$.

Ex. $W = S_\pi V$, highest weight is π. $\pi = (p_1, \ldots, p_{\mathbf{v}}) \Rightarrow$ $proj_{\pi-def}(e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_{\mathbf{v}}^{\otimes p_{\mathbf{v}}})$ hw vect.

Lie-Kolchin Thm: U G-module $\mathbb{B} \subset G$ Borel, $\exists$ $\mathbb{B}$-fixed line.

Moreover if U irred. G-module (G reductive) $\Rightarrow \exists!$ hw line.

$\mathbb{B}$-fixed spaces and border substitution

Borel: More generally X: projective G-variety, $\exists x \in X$, $\mathbb{B}$-fixed.

In border substitution method, can restrict to Borel fixed points in $G(\mathbf{a} - t, A^*)$:

Prop. (L-Michalek 2016) Let $T \in A \otimes B \otimes C$ be A-concise.

Then

$$\underline{\mathbf{R}}(T) \geq \min_{A' \subset A^*, \textbf{Borel fixed}, \operatorname{codim} A' = t} \underline{\mathbf{R}}(T|_{A' \otimes B^* \otimes C^*}) + t$$

Next step: Describe Borel fixed subspaces

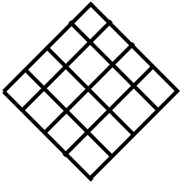
Weight diagrams

W: G-module has a basis consisting of weight vectors.

$\mathbb{B} \rightsquigarrow$ partial order on weights and weight lines. $[w] \leq [u]$ if $[u] \in \overline{\mathbb{B}[w]}$

$G = GL(V)$ get dominance order on sequences of integers $(p_1, \ldots, p_{\mathbf{v}})$. $\rightsquigarrow$ weight diagram.

Ex. $U^* \otimes V$ as $G = GL(U) \times GL(V)$-module, highest weight vect. $= u^{\mathbf{n}} \otimes v_1$ weight diagram:



Borel fixed subspaces (i.e., Borel fixed points of Grassmannian)

Easy to draw .

Borel subgroups in general

Given G have assoc. Lie algebra $\mathfrak{g} = T_{\mathrm{Id}} G$.

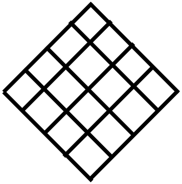
$\mathfrak{g}$ is *solvable* if, $D_1(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$, $D_k(\mathfrak{g}) := [D_{k-1}(\mathfrak{g}), D_{k-1}(\mathfrak{g})]$ is 0 some k.

G *solvable* if $\mathfrak{g}$ is solvable.

G: reductive, $\exists$ maximal solvable subgroups $=:$ Borel subgroups

Example $M_{\langle n \rangle}$

$A = U^* \otimes V$



Computation $\rightsquigarrow$ worst case is along one of outer diagonals.

$\rightsquigarrow$ Thm. (L-Michalek, 2018) $\underline{\mathbf{R}}(M_{\langle n \rangle}) \geq 2\mathbf{n}^2 - \lceil \log_2(\mathbf{n}) \rceil - 1.$

Hay in a haystack: explicit tensors of high border rank

Tension: want $\dim(G_T) > 0$ and T as "generic" as possible

So far best with $G_T = \mathbb{C}^*$:

$\forall$ k, define $\lceil \frac{3(2k+1)^2}{4} \rceil$-dimensional family
$T_k = T_k(p_{ij}) \in \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2k+1} =: A \otimes B \otimes C$, where
$|i|, |j|, |i+j| \leq k$ as follows:

$$T_k = \sum_{i=-k}^{k} \sum_{j=\max(-k,-i-k)}^{\min(k,-i+k)} p_{ij} a_i \otimes b_j \otimes c_{-i-j},$$

here $(a_{-k}, \dots, a_k), (b_{-k}, \dots, b_k), (c_{-k}, \dots, c_k)$ are bases.

Take p_{ij} distinct primes (computer science explicit) $\rightsquigarrow$
$\underline{\mathbf{R}}(T_k) \geq (2.02)(2k+1) = (2.02)m$ once $k \sim 10^8$. (L-Michalek 2019) (Already can beat $2m$ with smallish k.)

Game over?

Both cases limits of known methods. How to go further?

Buczynska-Buczynski idea: use more information limits of *ideals*
$I_t :=$ ideal of $[T_1(t)] \sqcup \cdots \sqcup [T_r(t)]$

Problem: How to take limits?

First Idea: The Hilbert scheme

Grothendiek: insist on saturated ideals $I \subset Sym(V^*)$. Then, in a sufficiently high degree D, $I_D \subset S^D V^*$ determines I in all degrees. and sufficiently high can be made precise. Reduced to taking limits in one fixed Grassmannian. "Hilbert Scheme"(Parametrizes saturated ideals with same Hilbert polynomial.)

$I \subset Sym(V^*)$ any ideal. Let $r_d = \dim(S^d V^*/I_d)$. (Hilbert function $h_I(d) := r_d$)

Fact: (Castelnuovo-Mumford regularity) if fix Hilbert function, $\exists$ explicit $D = D(h_I)$ such that I_D determines $I_{D'}$ for all $D' > D$.

Moreover $h_I(x)$ poly when $x > D \rightsquigarrow$ Hilbert polynomial

Hilbert scheme lives in $G(D, S^D V^*)$.

BB idea

If have border rank decomp. $T = \lim_{t\to 0} \sum_{j=1}^{r} T_j(t)$ can study $I_t :=$ ideal of $[T_1(t)] \sqcup \cdots \sqcup [T_r(t)]$

Bad news: Hilbert scheme doesn't work. Consider toy case of 3 points in $\mathbb{P}^2$: $[1,0,0],[0,1,0],[1,-1,t]$ $t \neq 0$, $(I_t)_1 = 0$ and

$(I_t)_2 = \langle x_3^2 + t^2 x_1x_2,\ x_3^2 - tx_1x_3,\ x_1x_3 + x_2x_3\rangle$

But $(I_0)_1 = \langle x_3\rangle$, $(I_0)_2 = \langle x_3^2, x_1x_3 + x_2x_3\rangle$

Ideal of limiting scheme in fixed deg. $\neq$ limit of spans

Limit is taken in Hilbert scheme (i.e. a fixed Grasssmannian), loses information important for border rank decomposition.

Solution: Haiman-Sturmfels multi-graded Hilbert scheme.

BB idea

Consider product of Grasmannians

$G(r_1, V^*) \times G(r_2, S^2 V^*) \times \cdots \times G(r_D, S^D V^*)$

and map $I \mapsto ([I_1] \times [I_2] \times \cdots \times [I_D])$. For each $\mathbb{Z}_{\geq 0}$-valued function h, get (possibly empty) subscheme parametrizing all ideals I with *Hilbert function* $h_I = h$.

Rigged such that limit I of ideals has same Hilbert function as ideals I_ϵ.

Key Lemma: In border rank decompositions, WLOG for $t > 0$ points in general position $\rightsquigarrow$ const. Hilbert function (as soon as possible)

BB idea: Tensor case

have more information: curves of points on $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$
can think of ideals in
$Sym(A \oplus B \oplus C)^* = \bigoplus_{s,t,u} S^s A^* \otimes S^t B^* \otimes S^u C^*$, $\mathbb{Z}^{\oplus 3}$-graded.

Hilbert function $h_I(s,t,u) := \dim(S^s A^* \otimes S^t B^* \otimes S^u C^* / I_{s,t,u})$.

Key Lemma $\Rightarrow$ Know Hilbert function! Namely
$h_I(s,t,u) = \min\{r, \dim S^s A^* \otimes S^t B^* \otimes S^u C^*\}$.

BB idea: Tensor case summary

Instead of single curve $E_t \subset G(r, A{\otimes}B{\otimes}C)$ limiting to Borel fixed point, for each (i, j, k) get curve in $G(r, S^i A^* {\otimes} S^j B^* {\otimes} S^k C^*)$, each limiting to Borel fixed point *and* satisfying compatibility conditions.

Upshot: algorithm that either produces all normalized candidate I_0's or proves border rank $> r$.

The border apolarity method

If $\underline{\mathbf{R}}(T) \leq r$, there exists a multi-graded ideal I satisfying:

1. I is contained in the annihilator of T. This condition says $I_{110} \subset T(C^*)^\perp$, $I_{101} \subset T(B^*)^\perp$, $I_{011} \subset T(A^*)^\perp$ and $I_{111} \subset T^\perp \subset A^* \otimes B^* \otimes C^*$.
2. For all (ijk) with $i + j + k > 1$, $\mathrm{codim} I_{ijk} = r$.
3. each I_{ijk} is Borel-fixed.
4. I is an ideal, so the multiplication maps $I_{i-1,j,k} \otimes A^* \oplus I_{i,j-1,k} \otimes B^* \oplus I_{i,j,k-1} \otimes C^* \to S^i A^* \otimes S^j B^* \otimes S^k C^*$ have image contained in I_{ijk}.

Next time: unpack these conditions and apply to matrix multiplication and other tensors.

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**: